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# Perturbation Theory for Infinite Dimensional Dynamical Systems

**Andrew Stuart**

*Program in Scientific Computing and Computational Mathematics,*
*Division of Applied Mechanics,*
*Durand 252,*
*Stanford University,*
*California 94305-4040,*
*USA* [1]

**Abstract**

When considering the effect of perturbations on initial value problems over long time intervals it is not possible, in general, to uniformly approximate individual trajectories. This is because well-posed initial value problems allow exponential divergence of trajectories and this fact is reflected in the error bound relating trajectories of the perturbed and unperturbed problems. In order to interpret data obtained from numerical simulations over long time intervals, and from other forms of perturbations, it is hence often necessary to ask different questions concerning the behavior as the approximation is refined. One possibility, which we concentrate on in this review, is to study the effect of perturbation on sets which are invariant under the evolution equation. Such sets include equilibria, periodic solutions, stable and unstable manifolds, phase portraits, inertial manifolds and attractors; they are crucial to the understanding of long-time dynamics.

An abstract semilinear evolution equation in a Hilbert space $X$ is considered, yielding a semigroup $S(t)$ acting on a subspace $V$ of $X$. A general class of perturbed semigroups $S^h(t)$ are considered which are $C^1$ close to $S(t)$ uniformly on bounded subsets of $V$ and time intervals $[t_1, t_2]$ with $0 < t_1 < t_2 < \infty$. A variety of perturbed problems are shown to satisfy these approximation properties. Examples include a Galerkin method based on the eigenfunctions of the linear part of the abstract sectorial evolution equation, a backward Euler approximation of the same equation and a singular perturbation of the Cahn–Hilliard equation arising from the phase-field model of phase transitions. The invariant sets of $S(t)$ and $S^h(t)$ are compared and convergence properties established.

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# 1 Introduction

The accumulation of errors caused by perturbation of a well-posed evolution equation is governed by the properties of the evolution equation itself. In regions where nearby solutions of the equation tend to diverge exponentially in time, the error introduced by perturbation will tend to grow exponentially in time. Consider, for example, the perturbation caused by numerical approximation of an evolution equation. Because of the exponential divergence of trajectories, typical a priori error estimates between true and numerical solutions contain error constants which grow as the exponential of the time interval under consideration. For many nonlinear problems this exponential divergence may not persist for all time and it is possible to obtain greatly reduced error constants by means of a posteriori error analysis (see Eriksson *et al.* [33]) whereby the error constant is estimated as the computation proceeds rather than being majorized by an exponential in time. It is nonetheless a fact that, for most problems and for both a priori and a posteriori error estimates, *it is not possible to approximate a true trajectory with a numerical trajectory starting from the same point, uniformly on an infinite time interval.* The only exception is the case where trajectories are contracting which, for autonomous problems, implies that the solution being approximated is asymptotic to a stable steady state. It is thus natural to ask how numerical data from long time simulations involving more complicated behavior than convergence to a steady state should be interpreted. Such long time simulations are of great importance in science and engineering arising, for example, in the simulation of turbulent fluid flow, in phase transition calculations and in interplanetary interactions. In all these examples it is commonplace to integrate past the time at which standard a priori or a posteriori estimates guarantee closeness of trajectories. Although we have talked primarily about the effect of perturbations introduced through numerical approximation, similar considerations are relevant, for example, in studying the effect of singular perturbations of the coefficients in evolution equations.

The properties of an evolution equation on a Hilbert space $V$ are captured by a semigroup $S(t)$ mapping an initial data point $v \in V$ to the solution of the equation at time $t$, namely $S(t)v$. The semigroup can also be extended to act on subsets of $V$ in the natural way. Of crucial importance to understanding the long time behavior of an evolution equation are the sets which remain invariant under $S(t)$. Such sets may include simple objects such as equilibria or periodic solutions and also more complicated sets such as those arising in chaotic systems; also of importance are invariant manifolds such as the stable and unstable manifolds of equilibrium points and the inertial manifold, a set on which the dynamics of certain partial differential equations is governed by ordinary differential equations. Here we study the effect of small perturbations to $S(t)$, such as those arising from numerical approximation or from singular perturbations to partial differential equations, on these invariant sets. As we shall see the analysis is considerably streamlined by assuming that the perturbation is $C^1$ close to $S(t)$, in a sense to be made precise. Such $C^1$ closeness results can be proved for many numerical methods and for singular perturbations of partial differential equations. They are are established in this article in a variety of contexts.

In Section 2 we introduce the class of evolution equations which we use to illustrate the analysis. These equations are formulated as ordinary differential equations in a Hilbert space and take the form of a linear differential operator (whose solutions decay in time) with a lower-order nonlinear perturbation. Examples which are considered include an abstract sectorial evolution equation, reaction-diffusion equations, some important fourth-order pattern formation problems and the Navier–Stokes equation. Existence, uniqueness and regularity results are described for the abstract sectorial evolution equation, using a variation of constants formula for the problem. Appropriate references are given to the analogous results for other classes of problems. Section 2 concludes with a summary of the important definitions and results from the theory of dynamical systems that will be useful to us here.

In Section 3 we state our basic assumptions concerning the perturbed semigroup $S^h(t)$ : it is $C^1$ close to the semigroup $S(t)$ generated by the underlying equation itself, uniformly in bounded subsets of $V$ and on compact time-intervals disjoint from the origin. We then describe various examples where this is satisfied. We initially consider the abstract sectorial evolution equation and introduce a spectral method based on the eigenfunctions of the linear differential operator. We prove standard error estimates for trajectories together with results concerning the effect of numerical approximation of the Fréchet derivative of the semigroup with respect to initial data. Throughout, the variation of constants formula is used to unify the presentation. Similar results are described for a singular perturbation of the Cahn–Hilliard equation, known as the viscous Cahn–Hilliard equation.

In Section 4 we consider the neighborhood of an equilibrium point. We first show that, provided $\bar{u}$ is a hyperbolic equilibrium point of $S(t)$, then there is a nearby equilibrium point $\bar{u}^h$ of the approximate semigroup $S^h(t)$. We then show that, if the solution being approximated approaches an exponentially attracting equilibrium point as $t \to \infty$, then it *can be uniformly approximated over infinite time intervals.* This situation is the only case where uniform-in-time approximation of individual trajectories is possible for autonomous problems.

Following this we construct a local phase portrait comprising the union of solutions of $u(t) = S(t)u(0)$ in the neighborhood of an equilibrium point $\bar{u}$ of saddle type. This local phase portrait is shown to persist under the basic approximation assumptions. Since these solutions are defined over arbitrarily long time intervals, a standard error estimate cannot be used directly in proving this result; instead we show that each solution of $S(t)$ may be approximated by a solution of $S^h(t)$ *starting from a different initial condition.* Understanding the effect of numerical perturbation on phase portraits is of use for the interpretation of data found when directly computing phase portraits close to equilibria. It is also useful as a building block for the proof of other results such as the uniform in time, piecewise continuous, error estimates for gradient systems which are described in Section 8.

In Section 5 we look at unstable manifolds of a hyperbolic equilibrium point $\bar{u}$. Locally in the neighborhood of $\bar{u}$ these may be constructed as part of the phase portrait discussed in Section 4; however, we describe an alternative presentation based on a technique known as a graph transform. Again, persistence of a nearby local unstable manifold under the assumptions on the perturbation is proved. Every point on the true local unstable manifold is close to a point on the approximate local unstable manifold and vice versa. Having studied local unstable manifolds the results are extended to the global unstable manifold by a compactness argument. There are two main reasons for studying unstable manifolds: they are of interest in their own right and, furthermore, are an important building block in the study of attractors — something pursued further in Section 7.

In Section 6 we study inertial manifolds. These objects are of theoretical importance in the study of a partial differential equation since they show that certain infinite-dimensional systems are governed by finite-dimensional systems for large time. Indeed, on its inertial manifold, the equation reduces to an ordinary differential equation known as an inertial form. It is thus of some interest to understand the effect of perturbation on the inertial manifold. The techniques we use are very similar to the graph transform techniques used in Section 5 for the construction of unstable manifolds. Under appropriate conditions we prove that $S(t)$ and $S^h(t)$ have inertial manifolds $\mathcal{M}$ and $\mathcal{M}^h$ and that every point of $\mathcal{M}$ is close to a point in $\mathcal{M}^h$ and vice versa.

In Section 7 we study attractors — these are sets which attract an open neighborhood of themselves under the evolution equation. Simple examples are stable equilibrium points and periodic solutions; however, the reason for abstracting to this general object is that in very complex problems (for example those exhibiting chaos) the most basic description of what we observe after a long time is contained in the notion of attractor. We assume that $S(t)$ and $S^h(t)$ have attractors $\mathcal{A}$ and $\mathcal{A}^h$ and then show that every point on $\mathcal{A}^h$ is close to a point on $\mathcal{A}$ for $h$ small. Unfortunately it is not possible to prove the converse in general; simple counter-examples exist to illustrate why. Thus parts of an attractor may disappear under small perturbations. However, we know by considering the simple example of an exponentially attracting stable equilibrium point that, for some problems, the whole attractor *will* be well approximated. The remainder of the chapter is devoted to studying two classes of attractor for which the whole object perturbs by a small amount under the basic assumptions about the perturbing semigroup. The first class is the class of attractors $\mathcal{A}$ and $\mathcal{A}^h$ which are uniformly exponentially attracting; the second class is the class of attractors made up of the union of unstable manifolds and in this connection we exploit the theory of Section 5.

In Section 8 we turn our attention to error estimates for gradient systems. These systems are characterized by a globally decreasing Lyapunov functional which drives solutions towards an equilibrium point for large time. Using this property and the results of Sections 3 and 4, it is possible to approximate uniformly almost all trajectories of gradient systems by solutions of the perturbed system. However, in so doing, the error constant behaves badly with respect to initial data. In particular, the error constant is not uniformly bounded as the initial data is varied in a bounded set. For this reason the value of the error estimate is severely diminished. To obviate this problem we introduce a weaker concept of piecewise approximation of trajectories: we seek approximation of true solutions on $(0, \infty)$ by a piecewise continuous trajectory of the approximate problem with a finite number of discontinuities. In this context we derive uniform in time error estimates with error constants bounded uniformly in a bounded set of initial data.

Note that the analysis outlined so far is primarily concerned with proving persistence of invariant sets, and their convergence, under a basic assumption about the perturbed semigroup which includes, but is not restricted to, a variety of numerical approximation techniques. As such no distinction is made between the practical value of different numerical methods for dynamical systems other than in their rate of convergence. In order to assess the practical value of various schemes for dynamical systems it is of value to generalize the idea of *practical numerical stability* to classes of nonlinear problems arising in practice. In Section 9 we briefly turn our attention to this question.

Certain themes persist throughout the article and important results, frequently used and not necessarily proved, have been summarized in the Appendices. The first recurrent theme is the use of a *variation of constants* approach to the evolution equations: we use it in Sections 2 and 3 to prove existence, regularity and error estimates for the abstract sectorial evolution equation and its spectral approximation. We also use it in Sections 4 and 5 to enable us to replace the differential equation in the neighborhood of an equilibrium point by a mapping which retains the "linear plus small nonlinear" structure inherent in such problems; similar considerations apply for the construction of inertial manifolds in Section 6. The use of mappings is motivated by a desire to prove results which apply to time-discrete as well as time-continuous perturbations. The essential material for understanding the variation of constants approach to evolution equations in a Hilbert space is given in Appendix A. The second recurrent theme is the use of *uniform contraction principles* to construct objects of interest (such as equilibria, phase portraits and invariant manifolds) and at the same time to incorporate the effect of perturbation in the analysis. Contraction principles of various types used throughout the article are described in Appendix B; the *Taylor expansion in a Hilbert space* is also given in that Appendix and used several times in the main body of the text. The third recurrent theme is *attractive invariant manifolds.* Simple examples are unstable manifolds and inertial manifolds and both of these are constructed by use of an abstract theorem concerning attractive invariant manifolds given in Appendix C.

Some of the material described here is also presented in the context of time-discretization of ordinary differential equations in $\mathbb{R}^m$ in Stuart [88]. However there are certain technically challenging difficulties which arise in the consideration of partial differential equations which mean that different or modified techniques need to be applied.

## 2 Evolution equations in a Hilbert space

### 2.1 Introduction

This section commences with an introduction to the basic properties of sectorial operators in a Hilbert space. The important material, together with some examples, is sketched. More details and references are given in Appendix A. This material is followed by a brief summary of some of the important notation used throughout the article. In Section 2.4 the nonlinear evolution equation which we study is introduced and the assumptions about its solution given. These basically amount to asking that the solution depend on time and initial data in a $C^1$ sense.

We then describe a variety of situations in which the assumptions hold. We concentrate on an abstract evolution equation in a Hilbert space, governed by a sectorial evolution equation. The primary result of importance here is Theorem 2.6 which describes the existence and regularity for the equation. The **Important remark** following the theorem introduces the use of a variation of constants approach to the nonlinear partial differential equation, a theme which recurs throughout. We also mention briefly the Navier–Stokes equation, the Cahn–Hilliard equation and ordinary differential equations, all of which fit into our framework.

The section concludes with a summary of some basic results in the theory of dynamical systems.

### 2.2 Sectorial operators

We consider the background theory of certain abstract evolution equations in a Hilbert space. For this we need the idea of a sectorial operator. For details concerning sectorial operators together with definitions of their fractional powers and exponentials see Appendix A. We will refer freely to results in this Appendix throughout the article, and the reader is encouraged to study it at this point.

The easiest context in which to understand sectorial operators is the theory of self-adjoint operators. Consider a separable Hilbert space $X$ with inner product $\langle \bullet, \bullet \rangle$ and norm defined by $| \bullet |^2 = \langle \bullet, \bullet \rangle$. We assume that $A$ is a closed, densely defined, self-adjoint, positive operator with compact inverse; we denote the associated eigenfunctions by $\{\varphi_i\}$ and the positive real eigenvalues by $\{\lambda_i\}$ with the ordering chosen so that

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \quad . \tag{2.1}$$

In the case primarily considered here, where $X$ is infinite-dimensional, we have $\lambda_n \to \infty$ as $n \to \infty$.

The properties of $A$ ensure that it is a sectorial operator so that we may define fractional powers of $A$, $A^\alpha$, and also the resulting Hilbert spaces $X^\alpha = D(A^\alpha)$ with norms $| \bullet |_\alpha = |A^\alpha \bullet |$. For each $\alpha > \beta \geq 0$ it follows that the inclusion $X^\alpha \subset X^\beta$ is compact. Furthermore, we may define the operator $e^{-At}$ for $t > 0$. Of particular interest to us is the space $X^\beta$ where $\beta \in [0, 1)$ appears in Assumption 2.3 and in (2.11) below. We employ the notation

$$V \equiv X^\beta \quad \text{and} \quad \| \bullet \| \equiv | \bullet |_\beta.$$

Initial data for our basic equation (2.9) will be specified in $V$.

**Example 2.1** Let $X = L_2(\Omega)$, $A = -\frac{d^2}{dx^2}$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $\Omega = (0, 1)$. Thus

$$(v, w) = \int_0^1 v(x)w(x)dx,$$

and $| \bullet |$ denotes the $L_2$ norm. Also $\varphi_j = \sqrt{2}\sin(j\pi x)$ and $\lambda_j = j^2\pi^2$ for $j = 1, 2, \dots, \infty$. It is well known that any function $v \in X$ can be written as

$$v = \sum_{j=1}^{\infty} v_j \varphi_j, \quad v_j = (v, \varphi_j) \tag{2.2}$$

where the $v_j \in \mathbb{R}, j = 1, 2, \dots, \infty$ satisfy

$$\sum_{j=1}^{\infty} v_j^2 < \infty.$$

The fractional powers of $A$ are generated by defining

$$A^\alpha v = \sum_{j=1}^{\infty} \lambda_j^\alpha v_j \varphi_j.$$

Thus

$$|v|_\alpha^2 = \sum_{j=1}^{\infty} \lambda_j^{2\alpha} v_j^2.$$

Roughly speaking $D(A^\alpha)$ may be thought of as the set of functions $v$ such that

$$\sum_{j=1}^{\infty} \lambda_j^{2\alpha} v_j^2 < \infty,$$

with $v_j$ given by (2.2). For example, if $\alpha = \frac{1}{2}$, we find that $X^{\frac{1}{2}} \equiv H_0^1(\Omega)$. This may be seen by noting that, formally,

$$\begin{aligned} |v|_{\frac{1}{2}}^2 = \sum_{j=1}^{\infty} j^2\pi^2 v_j^2 &= \langle \sum_{j=1}^{\infty} j\pi v_j \sqrt{2}\cos(j\pi x), \sum_{j=1}^{\infty} j\pi v_j \sqrt{2}\cos(j\pi x)\rangle \\ &= |\sum_{j=1}^{\infty} j\pi v_j \sqrt{2}\cos(j\pi x)|^2 = |v_x|^2. \end{aligned}$$

Recalling that $|v_x|$ is a norm on $H_0^1(\Omega)$ the result follows.

For future use we also note here the following basic norm inequalities:

$$\begin{aligned} |v| \leq \tfrac{1}{\pi}|v|_{\frac{1}{2}} \quad \forall v \in H_0^1(\Omega), \\ \|v\|_\infty := \sup_{x\in(0,1)} |v(x)| \leq |v|_{\frac{1}{2}} \quad \forall v \in H_0^1(\Omega). \end{aligned} \tag{2.3}$$

Note that the heat equation

$$u_t + Au = 0,\ u(0) = u_0 := \sum_{j=1}^{\infty} u_j \varphi_j$$

has solution

$$u(t) = e^{-At}u_0 = \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j \varphi_j;$$

this gives the appropriate way to think about exponentials of sectorial operators in the self-adjoint case.

It cannot be over-emphasized that the preceding example, and the straightforward calculations given therein, are very specific to the self-adjoint problem. To understand fully the concepts of fractional powers and exponentials of sectorial operators, the reader should pursue in detail the references given in Appendix A. In any case it will be beneficial to the reader to study Appendix A before proceeding.

### 2.3 Notation

In subsequent sections we will need an appropriate definition of the distance between sets in $V$. Thus we introduce the following notation:

$$\left\{\begin{aligned} \text{dist}(u, A) &= \inf_{v\in A} \|u - v\|, \\ \text{dist}(B, A) &= \sup_{u\in B} \text{dist}(u, A), \\ \mathcal{N}(A, \varepsilon) &= \{u \in V : \text{dist}(u, A) < \varepsilon\}, \\ \partial\mathcal{N}(A, \varepsilon) &= \{u \in V : \text{dist}(u, A) = \varepsilon\}. \end{aligned}\right\} \tag{2.4}$$

It follows from (2.4) that, if $\text{dist}(B, A) < \varepsilon$, then $\bar{B} \subseteq \mathcal{N}(\bar{A}, \varepsilon)$. Hence

$$\text{dist}(B, A) = 0 \;\Leftrightarrow\; \bar{B} \subseteq \bar{A}.$$

Thus "dist" only defines a semidistance — the asymmetric *Hausdorff semi-distance.* The *Hausdorff distance* between two sets $A$ and $B$ is defined by

$$d_{\text{H}}(B, A) = \max\{\text{dist}(A, B), \text{dist}(B, A)\}. \tag{2.5}$$

Thus

$$d_{\text{H}}(A, B) = 0 \;\Leftrightarrow\; \bar{B} \equiv \bar{A}.$$

We also employ the notation

$$\begin{aligned} B(v, \varepsilon) &:= \{u \in V : \|u - v\| < \varepsilon\}, \\ \partial B(v, \varepsilon) &:= \{u \in V : \|u - v\| = \varepsilon\}. \end{aligned} \tag{2.6}$$

Thus $B(v, \varepsilon) = \mathcal{N}(v, \varepsilon)$ and $\partial B(v, \varepsilon) = \partial\mathcal{N}(v, \varepsilon)$. With this notation we make a definition.

**Definition 2.2** *A family of sets $\mathcal{A}^h$, $h \in [0, h_c]$, is termed upper semi-continuous at $h = 0$ if dist$(\mathcal{A}^h, \mathcal{A}_0) \to 0$ as $h \to 0$. The family is termed lower semi-continuous at $h = 0$ if dist$(\mathcal{A}_0, \mathcal{A}^h) \to 0$ as $h \to 0$. The family is continuous at $h = 0$ if it is both upper semi-continuous and lower semi-continuous at $h = 0$ so that $d_{\text{H}}(\mathcal{A}_0, \mathcal{A}^h) \to 0$ as $h \to 0$.*

In the remainder of the article we use the standard induced operator norm on linear mappings $L \in \mathcal{L}(V, V)$ namely

$$\|L\| := \sup_{\|\xi\|=1} \|L\xi\|. \tag{2.7}$$

We define the operator norm of $L \in \mathcal{L}(X, X)$ by

$$|L| := \sup_{|\xi|=1} |L\xi|. \tag{2.8}$$

### 2.4 The evolution equation

Throughout this article we study the behavior of the abstract evolution equation

$$\frac{du}{dt} + Au = F(u), t > 0, \quad u(0) = u_0. \tag{2.9}$$

The operator $A$ is assumed to satisfy the properties given in Section 2.1: it is a linear, closed, densely defined, self-adjoint, positive operator with compact inverse. The function $F$ is a continuous nonlinear operator from $X^\zeta$ to $X$, for some $\zeta \in [0, 1)$, whose properties are specified below. Its Fréchet derivative is denoted by $dF(\bullet)$. The precise definition of a solution to eqn (2.9) may be found in Lemma 10.8. Throughout this article we make the following assumptions concerning solutions of this equation.

**Assumption 2.3** *There is $\beta \in [0, 1)$ such that, for every $u_0 \in V \equiv X^\beta$, eqn (2.9) has a unique solution $u(t; u_0) \in V$ defined for $t \in [0, \infty)$. We denote by $S(t) : V \mapsto V$ the operator defined by*

$$S(t)v := u(t; v).$$

*There exists $\eta > \beta$ such that, for every $t > 0$ and every $v \in V$, $S(t)v \in X^\eta$. Furthermore, we assume that $S(\bullet)\bullet \in C^1(\mathbb{R}^+ \times V, V)$.*

Briefly, this assumption implies existence and uniqueness of a solution, continuous dependence upon the data, and a compactness property for the solution operator. *This assumption is made throughout without being explicitly stated in the results in the remainder of the article.*

We shall use the notation $dS(v; t)$ to denote the Fréchet derivative of $S(t)u$ with respect to $u$, evaluated at a point $v$. Note that, since the solution operator $S(t)\bullet$ is $C^1$ we deduce the following result.

**Lemma 2.4** *For any $R > 0$ there is a positive and increasing function $C(t)$, defined on $\mathbb{R}^+$, such that*

$$\|S(t)u - S(t)v\| \le C(t)\|u - v\| \quad \forall u, v \in B(0, R). \tag{2.10}$$

In Sections 2.5, 2.6, 2.7 and 2.8 we consider various examples illustrating that Assumption 2.3 is satisfied for a wide variety of problems.

### 2.5 Sectorial evolution equations

As our first example of a class of problems satisfying Assumptions 2.3 we consider (2.9) as an ordinary differential equation in the Hilbert space $X$. We make the following assumption concerning the function $F(u)$ appearing in (2.9). There exists a constant $K > 0$ and $\beta \in [0, 1)$ such that, for all

$u, v, w, \in V$ :

$$\begin{aligned} F &\in C^1(V, X); \\ |F(u)| &\leq K; \\ |F(u) - F(v)| &\leq K|u - v|_\beta; \\ |dF(u)v| &\leq K|v|_\beta; \\ |dF(u)w - dF(v)w| &\leq K|u - v|_\beta |w|_\beta. \end{aligned} \tag{2.11}$$

(Recall that $V \equiv X^\beta$ and that $\|\bullet\| \equiv |\bullet|_\beta$.) Under this assumption, and the standing assumptions on the operator $A$, we will prove that Assumption 2.3 holds. We start with an example illustrating (2.11).

**Example 2.5** As a first example consider the scalar reaction-diffusion equation

$$\begin{aligned} u_t = u_{xx} + f(u), \quad (x, t) \in (0, 1) \times (0, \infty), \\ u(0, t) = u(1, t) = 0, \quad t > 0, \\ u(x, 0) = u_0(x). \end{aligned} \tag{2.12}$$

We establish that (2.11) holds for this problem, under the assumption that $f \in C^\infty(\mathbb{R}, \mathbb{R})$ and that there exists a constant $C > 0$ such that

$$|f(u)| + |f'(u)| + |f''(u)| \leq C \quad \forall u \in \mathbb{R}. \tag{2.13}$$

Here $A$ and $D(A)$ may be defined as in Example 2.1; recall that $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$. We define $F(u)$ by

$$F(u)(x) := f(u(x))$$

and then

$$(dF(u)v)(x) = f'(u(x))v(x).$$

That $F$ is $C^2$ follows from the smoothness of $f$. First note that

$$|F(u)|^2 = \int_0^1 f(u(x))^2 dx \leq C^2.$$

Secondly note that, if $u, v \in H_0^1(\Omega)$, then $u$ and $v$ are bounded pointwise by (2.3); thus $f'(\xi(x))$ is bounded pointwise if $\xi(x) = su(x) + (1 - s)v(x)$ for some $s = s(x) \in [0, 1]$. By the mean value theorem and (2.3) it follows that

$$|F(u) - F(v)|^2 = \int_0^1 [f(u(x)) - f(v(x))]^2 dx$$

$$\leq \int_0^1 [f'(\xi(x))]^2(u(x)-v(x))^2dx$$
$$\leq C^2|u-v|^2 \leq C^2|u-v|^2_{\frac{1}{2}}/\pi^2.$$

Thirdly we have that, by (2.3),

$$|dF(u)v|^2 = \int_0^1 [f'(u(x))]^2 v(x)^2 dx \leq C^2|v|^2 \leq C^2|v|^2_{\frac{1}{2}}/\pi^2.$$

Finally note that, by arguments similar to those used in bounding $|F(u) - F(v)|$, we have

$$\begin{aligned}|dF(u)w - dF(v)w|^2 &= \int_0^1 [f'(u(x)) - f'(v(x))]^2 w^2(x)dx \\ &\leq \int_0^1 f''(\eta(x))^2(u(x)-v(x))^2 w(x)^2 dx \\ &\leq C^2\|u-v\|^2_\infty |w|^2 \\ &\leq C^2|u-v|^2_{\frac{1}{2}}|w|^2_{\frac{1}{2}}/\pi^2.\end{aligned}$$

Thus (2.11) holds with $\beta = 1/2$. The same result may be established in dimensions 2 and 3 by a slightly more subtle analysis.

As a second example consider the equation

$$\begin{gathered} u_t = -u_{xxxx} + f(u), \quad (x,t) \in (0,1) \times (0,\infty), \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x). \end{gathered}$$

This is sometimes known as the Swift–Hohenberg equation. Let $A_0$ denote the operator denoted by $A$ in Example 2.1 and now set $A = A_0^2$; the eigenvalues of $A$ are $j^4\pi^4$ and now $|v_x|^2 = |v|^2_{\frac{1}{4}}$. We make the same assumptions about $f$ as for the reaction-diffusion equation and define $F$ in the same way. Consequently, by following the analysis of the previous example, we have that (2.11) holds with $\beta = 1/4$.

We now prove the following result.

**Theorem 2.6** *Let (2.11) hold. Then, for every $u_0 \in V$, there exists a unique solution $u(t)$ of (2.9). Furthermore, there exist constants $C_1 = C_1(T,R)$ and $C_2 = C_2(\alpha,T,R) > 0$, such that, for all $t \in (0,T)$ and $u_0 \in B(0,R)$:*

$$\begin{aligned} |u(t)|_\alpha &\leq \frac{C_1}{t^{\alpha-\beta}}, \quad \forall \alpha \in [\beta, 1]; \\ \left|\frac{du}{dt}(t)\right|_\alpha &\leq \frac{C_2}{t^{\alpha-\beta+1}}, \quad \forall \alpha \in [\beta-1, 1). \end{aligned} \tag{2.14}$$

*Finally the solution operator* $S(t)u_0 := u(t)$ *satisfies Assumptions* 2.3.

**Proof** The existence of a solution globally defined in $t > 0$ follows from Theorem 10.9, using (2.11) to establish (10.6) and (10.9). The bound on $du/dt$ follows directly from the estimate in Theorem 10.9. Now let $\alpha \in [\beta, 1]$. Note that from (2.9),

$$|u|_\alpha = |A^{\alpha-1}Au| \leq \left|\frac{du}{dt}\right|_{\alpha-1} + |F(u)|_{\alpha-1}.$$

By Lemma 10.6, $A^{\alpha-1}$ is bounded on $X$ for $\alpha \leq 1$ and so, by (2.11) and the bound on $du/dt$ we obtain

$$|u|_\alpha \leq \frac{K_1}{t^{\alpha-\beta}} + K_2|F(u)| \leq \frac{K_1}{t^{\alpha-\beta}} + K_2 K \leq \frac{C(T)}{t^{\alpha-\beta}},$$

the required result on $|u|_\alpha$ for $\alpha \in [\beta, 1]$. It follows that $S(\bullet)\bullet$ is in $C^1(\mathbb{R}^+ \times V, V)$ by Theorem 10.10, since $F \in C^1(V, X)$ by (2.11). This completes the proof. □

**Important remark** Since the proof of the bounds in Theorem 2.6 follow directly from the stated bound on $du/dt$ in Theorem 10.9 and are hence somewhat obscure to a reader unfamiliar with Henry [54] or Pazy [81], we sketch a direct proof of the bounds on $u$ which is valid for $\alpha \in [\beta, 1)$. This introduces an approach to the analysis of (2.9) and its applications, based on the variation of constants formula, that will be useful to us in a variety of contexts. By formally using $e^{-At}$ as an integrating factor and noting that it is the solution operator for the linear problem (10.7), it follows that $u(t)$ satisfies

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u(s))ds. \tag{2.15}$$

Applying $A^\alpha$ to (2.15) we obtain

$$A^\alpha u(t) = A^\alpha e^{-At}u_0 + \int_0^t A^\alpha e^{-A(t-s)}F(u(s))ds.$$

Noting that fractional powers of $A$ commute with $e^{-At}$ we obtain

$$|u(t)|_\alpha \leq |A^{\alpha-\beta}e^{-At}A^\beta u_0| + \int_0^t |A^\alpha e^{-A(t-s)}|\,|F(u(s))|ds.$$

Applying Lemma 10.6, (2.11) and (10.4), we obtain

$$|u(t)|_\alpha \leq \frac{C_1}{t^{\alpha-\beta}}\|u_0\| + \int_0^1 \frac{C_2}{(t-s)^\alpha}ds.$$

The second term is integrable and proportional to $(1-\alpha)^{-1}$ if $\alpha < 1$ and the bound on $|u(t)|_\alpha$ follows for $\alpha \in [\beta, 1)$.

It is also of interest to prove the Lipschitz property (2.10) without appealing to the abstract Theorem 10.10. In fact, under (2.11), we have a global Lipschitz property. From (2.15), Lemma 10.6 and (2.11) we see that

$$\|S(t)u - S(t)v\| \leq \|u - v\| + \int_0^t \frac{CK}{(t-s)^\beta}\|S(s)u - S(s)v\|ds.$$

Applying the Gronwall Lemma 10.11 we obtain

$$\exists C = C(t) > 0 : \|S(t)u - S(t)v\| \leq C\|u - v\| \quad \forall u, v \in V. \tag{2.16}$$

Without loss of generality we may assume that $C(t)$ is bounded as $t \to 0$ and that it is monotonically increasing in $t$.

Note that by Theorem 2.6 the solution of (2.9) subject to (2.11) is a $C^1$ function of time $t$ and the initial data $u_0$. This allows us to consider the concept of the derivative of the solution with respect to initial data. We can now try and find the equation which the derivative satisfies. This can be calculated by linearizing eqn (2.9) about a given solution with initial data $u_0$. Setting

$$w(t) = u(t) + \eta v(t),$$

where $u(t)$ and $w(t)$ both satisfy (2.9) and $\eta \in \mathbb{R}$, we find that

$$\frac{du}{dt} + Au = F(u), \quad u(0) = u_0,$$
$$\frac{dw}{dt} + Aw = F(w), \quad w(0) = w_0.$$

Hence $v(t)$ satisfies

$$\eta[\frac{dv}{dt} + Av] = F(u + \eta v) - F(u), \quad v(0) = \xi.$$

Linearizing so that $F(u + \eta v) \approx F(u) + \eta dF(u)v + \mathcal{O}(\eta^2)$ and letting $\eta \to 0$ gives the equation

$$\frac{dv}{dt} + Av = dF(u)v, \ t > 0, \quad v(0) = \xi, \tag{2.17}$$

where $u = u(t)$ solves (2.9). Thus $v(t)$ is the function found by calculating the derivative of the solution $u(t)$ with respect to initial data $u_0$ and applying it to $\xi$. For the following theorem, concerning the existence and regularity of the solution to (2.17), we need the concept of mild solution given in Lemma 10.7.

**Theorem 2.7** *Let (2.11) hold. For every $\xi \in V$ there exists a mild solution $v(t)$ of (2.17). Furthermore, there exists $C = C(T) > 0$ such that, for* *all* $t \in (0,T)$, $u_0 \in V$:

$$\|v(t)\| \leq C\|\xi\|;$$
$$|v(t)|_\alpha \leq \frac{C\|\xi\|}{(1-\alpha)t^{\alpha-\beta}}, \quad \forall \alpha \in (\beta, 1). \tag{2.18}$$

*Finally, if* $v(t) = dS(u_0,t)\xi$ *then* $v(t)$ *satisfies (2.17).*

**Proof** We employ Theorem 10.10 to deduce that (2.17) has a mild solution and that it represents the action of the operator $dS(u_0,t)$ on $\xi$. Thus the solution satisfies an integral equation obtained by the variation of constants formula

$$v(t) = e^{-At}\xi + \int_0^t e^{-A(t-s)} dF(u(s))v(s)ds. \tag{2.19}$$

Hence

$$A^\gamma v(t) = A^{\gamma-\beta}e^{-At}A^\beta\xi + \int_0^t A^\gamma e^{-A(t-s)} dF(u(s))v(s)ds.$$

Taking norms and applying Lemma 10.6 we obtain

$$|v(t)|_\gamma \leq \frac{C_1}{t^{\gamma-\beta}}\|\xi\| + \int_0^t \frac{C_2}{(t-s)^\gamma}\|v(s)\|ds. \tag{2.20}$$

Letting $\gamma = \beta$ and applying the Gronwall Lemma 10.11 we obtain the first result. Having obtained this we return to (2.20) with $\gamma = \alpha \in (\beta, 1)$ and integrate to obtain the second result:

$$\begin{aligned} |v(t)|_\alpha &\leq \frac{C_1\|\xi\|}{t^{\alpha-\beta}} + \int_0^t \frac{C_2 C\|\xi\|}{(t-s)^\alpha} ds \\ &= \frac{C_1\|\xi\|}{t^{\alpha-\beta}} + \frac{t^{1-\alpha}C_2C\|\xi\|}{(1-\alpha)} \\ &\leq \frac{C_1 + C_2CT^{1-\beta}}{(1-\alpha)t^{\alpha-\beta}}\|\xi\|. \end{aligned}$$

This completes the proof. □

**Important remark** At the expense of some unwieldy calculation it is possible to obtain greater regularity on $v(t)$ than that given here. However, in the context in which we are interested when considering the spectral approximation of (2.9), for example, the fact that the numerical solution converges in a $C^1$ sense is all that we need and the rate of convergence is not required. The rate of convergence of the spectral method is governed by the regularity of the solution being approximated and hence, for our purposes, it is not necessary to derive greater regularity on $v(t)$.

### 2.6 The Navier–Stokes equations

These equations may be written as

$$u_t + u \cdot \nabla u = \frac{1}{R}\Delta u - \nabla p + h(x), \quad x \in \Omega,$$

$$\nabla \cdot u = 0, \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial\Omega.$$

Let $X$ be the Hilbert space $\mathcal{H}$ comprising divergence free velocity fields contained in the space $L_2(\Omega)^2$ — see, for example, Temam [91]. This can be used to formulate an abstract evolution equation of the form (2.9) where $A$ denotes the Stokes operator and $F(u)$ comprises the effect of convection and the body forcing $h$. Then in two dimensions it may be shown that

$$|F(u) - F(v)| \leq K(R)|u - v|_\beta, \quad \forall u, v \in B(0, R)$$

provided that $\beta > \frac{1}{2}$. See, for example, Hale [50]. Thus (2.11) is not satisfied directly. However, under suitable conditions on $\Omega$, a priori bounds on the solution enable the construction of a modified $F$ which satisfies (2.11) and yields a problem which is equivalent to (2.21) for sufficiently large time. See Temam [92].

An alternative approach to the existence theory is to use the Faedo-Galerkin approach as described in Temam [91] and in Constantin and Foias [19]. This approach can be used to deduce that Assumptions 2.3 are satisfied with $X = V = \mathcal{H}$.

### 2.7 The Cahn–Hilliard equation

The Cahn–Hilliard equation (see Elliott [28] and the references therein) subject to Dirichlet boundary conditions may be written in the form

$$u_t = \Delta w, \quad x \in \Omega,$$

$$0 = \Delta u + f(u) + w, \quad x \in \Omega,$$

$$u = w = 0, \quad x \in \partial\Omega,$$

$$u = u_0(x), \quad t = 0.$$

Here a typical choice for $f(u)$ arising in applications is $f(u) = \beta(u - u^3)$ for some $\beta > 0$. Equation (2.21) may be formulated as an abstract evolution equation in the form of (2.9) by setting

$$u_t + A_0^2 u = A_0 F(u), \quad u(0) = u_0, \tag{2.21}$$

where $A_0$ is the operator denoted by $A$ in Example 2.1, and letting

$$F(u)(x) := f(u(x)).$$

The existence theory described in Section 2.5 and in Appendix A does not apply directly to this equation. However a similar theory can be developed and the Assumptions 2.3 shown to hold — see Elliott and Larsson [30] and Elliott and Stuart [32] for details.

## 2.8 Ordinary differential equations

As a final example of a system satisfying Assumptions 2.3, consider the system of ordinary differential equations of the form

$$u_t = f(u), \quad u(0) = u_0. \tag{2.22}$$

Assume that $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ and that structure is imposed on the function $f(u)$ ensuring global existence in time for all initial data in $\mathbb{R}^m$. Then solutions of eqn (2.22) are readily seen to generate a semigroup $S(t) : \mathbb{R}^m \mapsto \mathbb{R}^m$ such that Assumptions 2.3 hold with $V = \mathbb{R}^m$.

## 2.9 Semigroups

By Assumption 2.3 we know that a unique solution of (2.9) exists for all $t \geq 0$ and any $u_0 \in V$ and we have encountered several examples of classes of equations which do satisfy these assumptions. We have defined a semigroup $S(t) : V \to V$ in such a way that the solution $u(t)$ of (2.9) is given by

$$u(t) = S(t)u_0.$$

The one parameter mapping $S(t)$ satisfies the usual semigroup properties

(1) $S(0) = I$, the identity on $V$;

(2) $S(t + s) = S(t)S(s) \quad \forall t, s \in \mathbb{R}^+$.

For certain $u_0$ the operator $S(t)$ may be defined for $t < 0$; in such instances we will freely use $S(t)$ with negative arguments. A simple example is when $u_0$ is an equilibrium point. We now give some basic definitions and results concerning dynamical systems.

**Definition 2.8** *The action of $S(t)$ on a set $E \subset V$ is defined by*

$$S(t)E = \bigcup_{x \in E} S(t)x. \tag{2.23}$$

*A set $E$ is said to be invariant (resp. positively invariant, negatively invariant) if, for any $t \geq 0$, $S(t)E \equiv E$ (resp. $S(t)E \subseteq E$, $S(t)E \supseteq E$).*

**Example 2.9** The simplest example of an invariant set is an equilibrium point $\bar{u} \in D(A) \subset V$ satisfying $A\bar{u} = F(\bar{u})$ (see (4.13)). Since $du/dt \equiv 0$ if $u_0 = \bar{u}$ in (2.9) it is clear that $\bar{u}$ is invariant. Another simple example of an invariant set is a periodic solution of the equation.

To construct a simple positive invariant set assume that, for some $r > 0$,

$$\frac{d}{dt}\|u(t)\|^2|_{t=0} < 0 \quad \forall u_0 \in \partial B(0,r),$$

where $B(0,r)$ and $\partial B(0,r)$ are defined in (2.6).

Then no solutions starting in the set $B(0,r)$ can leave and hence it is positively invariant.

When comparing $S(t)$ and the perturbed semigroup $S^h(t)$ it is thus natural to compare the effect of approximation on the invariant sets (or positively invariant sets or negatively invariant sets) of $S(t)$. This is the approach we take in the remainder of the article. In general, given an initial data point $u_0$, we define the *forward orbit* to be $\{S(t)u_0, t \geq 0\}$. If there is $\varphi : (-\infty, 0] \to V$ with $\varphi(0) = u_0$ and $S(t)\varphi(s) = \varphi(t+s)$ for $0 \leq t \leq -s$ then a *negative orbit* of $u_0$ is $\{\varphi(t), t \leq 0\}$. This orbit will not exist for general $u_0$; when it does exist it may not be unique. If a negative orbit does exist then a *complete orbit* is the union of the positive and a negative orbits. The notion of backward orbits is very useful in the study of unstable manifolds (see Chapter 5). The notion of complete orbits will also be particularly useful in the study of attractors (see Chapter 7). A forward orbit is a simple example of a positively invariant set; a negative orbit is a simple example of a backward invariant set; a complete orbit is a simple example of an invariant set. Indeed it is a simple exercise to show that every point in an invariant set lies on a complete orbit.

It is important to be able to capture all the possible behavior of a dynamical system for large time. This is the motivation behind the following definition which leads to the identification of some further important invariant sets:

**Definition 2.10** *The $\omega$-limit set of a point $u_0$ is defined by*

$$\omega(u_0) = \{x \in V | \exists \{t_i\}, t_i \to \infty : S(t_i)u_0 \to x \text{ as } t_i \to \infty\}.$$

*An equivalent definition is*

$$\omega(u_0) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)u_0}. \tag{2.24}$$

*Similary we may define the $\omega$-limit set of a set $E \subset V$ by*

$$\omega(E) = \{x \in V | \exists \{t_i\}, \{u_i\}, t_i \to \infty, u_i \in E : S(t_i)u_i \to x \text{ as } t_i \to \infty\}.$$

*An equivalent definition is*

$$\omega(E) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)E}. \tag{2.25}$$

*Given a particular negative orbit through $u_0$, say $\{\varphi(t)|t \leq 0\}$, we define its $\alpha$-limit set by*

$$\alpha(u_0) = \{x \in V|\exists\{t_i\}, t_i \to -\infty : \varphi(t_i) \to x \text{ as } t_i \to -\infty\}.$$

Examples of $\omega$-limit and $\alpha$-limit sets of individual points are equilibrium points, periodic solutions, quasi-periodic solutions and more complicated objects such as the strange attractors observed in chaotic systems like the Lorenz equations. Note that, in general, we only have

$$\bigcup_{x \in E} \omega(x) \subset \omega(E).$$

Thus the $\omega$-limit sets of sets may be more complicated than simply the union of limit sets of individual trajectories — they also contain *heteroclinic and homoclinic orbits* connecting individual limit sets of trajectories. These are complete orbits with the same $\alpha$- and $\omega$-limit sets in the homoclinic case and differing ones in the heteroclinic case.

The following property of limit sets is fundamental.

**Theorem 2.11** *Assume that $E \subset V$ is non-empty and that there exists $t_0 \geq 0$ such that $\bigcup_{t \geq t_0} S(t)E$ is relatively compact. Then $\omega(E)$ is non-empty, compact and invariant. Furthermore, for any point $u_0 \in V$, $\omega(u_0)$ is connected.*

*For any negative orbit $\{\varphi(t), t \leq 0\}$ through $u_0$ for which there exists $t_1 \leq 0$ such that $\bigcup_{t \leq t_1} \varphi(t)$ is relatively compact, $\alpha(u_0)$ is non-empty, compact, invariant and connected.*

**Proof** Note that

$$\omega(E) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)E} = \bigcap_{s \geq t_0} \overline{\bigcup_{t \geq s} S(t)E}.$$

By the assumptions of the theorem, this is an intersection of nested non-empty compact sets; it is therefore non-empty and compact.

Now we show positive invariance: assume that $x \in \omega(E)$. If $S(t_i)v_i \to x$ then by continuity of $S(t)\bullet$,

$$S(t + t_i)v_i = S(t)S(t_i)v_i \to S(t)x \quad \forall t \geq 0.$$

Thus $S(t + t_i)v_i \to S(t)x$ and, hence, $S(t)x \in \omega(E)$ by Definition 2.10. Thus we deduce positive invariance of $\omega(E)$.

Now we establish negative invariance: assume that $x \in \omega(E)$. We wish to show that, for any $t > 0$, $\exists y$: $S(t)y = x$ and $y \in \omega(E)$. Let $S(t_i)v_i \to x$, where, without loss of generality, we may choose $t_1 \geq 1 + t_0 + t$. Now consider the sequence $S(t_i - t)v_i$. Since $\bigcup_{t \geq t_0} S(t)E$ is relatively compact it follows that there exists a convergent subsequence

$$S(t_{i_j} - t)v_{i_j} \to y.$$

Now

$$x = \lim_{j\to\infty} S(t_{i_j})v_{i_j} = \lim_{j\to\infty} S(t)S(t_{i_j} - t)v_{ij}$$

$$= S(t) \lim_{j\to\infty} S(t_{i_j} - t)v_{i_j} = S(t)y.$$

Hence the result is proved.

Finally we show that $\omega(u_0)$ is connected. Assume for contradiction that $\omega(u_0)$ has two disjoint components $P$ and $Q$ with $\mathcal{N}(P,\varepsilon) \cap \mathcal{N}(Q,\varepsilon) = \emptyset$, for some $\varepsilon > 0$. Then there exist sequences $t_i \to \infty$ and $\tau_i \to \infty$ such that $S(t_i)u_0 \to x \in P$ and $S(\tau_i)u_0 \to y \in Q$. Without loss of generality we may assume that $t_i < \tau_i$ and that $S(t_i)u_0 \in \mathcal{N}(P,\varepsilon)$, $S(\tau_i)u_0 \in \mathcal{N}(Q,\varepsilon)$ $\forall i \geq 1$. By continuity of $S(\bullet)u_0$ it follows that there exists $T_i \in (t_i, \tau_i)$ such that $S(T_i)u_0 \in \partial\mathcal{N}(P,\varepsilon)$. But the set $\partial\mathcal{N}(P,\varepsilon)$ is compact, since $P$ is compact. Thus there exists a convergent subsequence $S(T_{i_j})u_0 \to z$, where $z \in \partial\mathcal{N}(P,\varepsilon)$. But this is a contradiction since then, by definition, $z \in \omega(u_0)$ but $z \notin P \cup Q$. This completes the proof. The statements about $\alpha$-limit sets follow similarly. □

We conclude with three categories of dynamical systems which will be useful to us throughout this article.

**Definition 2.12** *The semigroup $S(t) : V \mapsto V$ is said to be contractive in the neighborhood of an equilibrium point $\bar{u}$ if there exist constants $\alpha, \sigma > 0$ such that, if $u_1, u_2 \in B(\bar{u}, \sigma)$, then*

$$\|S(t)u_1 - S(t)u_2\| \leq e^{-\alpha t}\|u_1 - u_2\| \quad \forall t \geq \tau.$$

In Section 4.3 we will prove that hyperbolic, stable equilibrium points yield contractive semigroups under certain natural conditions on the nonlinearity.

**Definition 2.13** *The semigroup $S(t) : V \mapsto V$ is said to be dissipative if there is a bounded set $B \subset V$, known as an absorbing set, such that for any bounded set $E \subset V$ there is $T = T(E, B)$ such that $S(t)E \subseteq B$ for all $t \geq T$.*

**Example 2.14** Consider eqn (2.12) under (2.13). By using the techniques outlined in Temam [92] it is possible to show that the equation generates a semigroup $S(t) : X \mapsto X$, where $X = L_2(0,1)$. Let $\langle\bullet,\bullet\rangle$ and $|\bullet|$ denote the inner product and norm on $X$. To see that the equation is dissipative on $X$ note that, by (2.3), for any $\epsilon > 0$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u|^2 &= \langle u, u_t\rangle \\ &= \langle u, u_{xx}\rangle + \langle u, f(u)\rangle \\ &\le -|u_x|^2 + \frac{\varepsilon^2}{2}|u|^2 + \frac{1}{2\varepsilon^2}|f(u)|^2 \\ &\le -\pi^2|u|^2 + \frac{\varepsilon^2}{2}|u|^2 + \frac{C^2}{2\varepsilon^2}.\end{aligned}$$

By choosing $\epsilon = \pi$ we have

$$\frac{d}{dt}|u|^2 \le \frac{C^2}{\pi^2} - \pi^2|u|^2.$$

Integration shows that the equation is dissipative on $X$ with absorbing set $B = B(0,\rho)$ for any $\rho : \rho^2 > C^2/\pi^4$.

**Definition 2.15** *The $C^1$ semigroup $S(t)$ generated by (2.9) is said to define a gradient system if there exists $\mathcal{V} \in C(V, \mathbb{R})$, called a Lyapunov function, satisfying*

*(i) $\mathcal{V}(u) \ge 0$ for all $u \in V$;*

*(ii) $\mathcal{V}(u) \to \infty$ as $\|u\| \to \infty$;*

*(iii) $\mathcal{V}(S(t)u)$ is nonincreasing in $t$ for each $u \in V$;*

*(iv) if $u$ is such that $S(t)u$ is defined for all $t \in \mathbb{R}$ and $\mathcal{V}(S(t)u) = \mathcal{V}(u)$ for $t \in \mathbb{R}$ then $u$ is an equilibrium point satisfying $Au = F(u)$.*

**Example 2.16** Consider the reaction-diffusion equation (2.12) subject to (2.13). By Theorem 2.6 this forms a dynamical system on $V \equiv H_0^1(\Omega)$. (Note that, in contrast, we considered the same equation as a dynamical system in $L_2((0,1))$ in Example 2.14.) If we define $h(u)$ to be a primitive of $f(u)$ so that $h'(u) = f(u)$ and set

$$\mathcal{V}(\varphi) := \int_0^1 \{\frac{1}{2}\varphi_x^2 - h(\varphi)\}dx$$

then $\mathcal{V}(\bullet)$ is a Lyapunov functional for (2.12) and hence we have a gradient system. To see this note that

$$h(u) = c + \int_0^u f(w)dw \tag{2.26}$$

for some arbitrarily chosen $c \in \mathbb{R}$. Thus, by (2.13), for any $\varepsilon > 0$,

$$\int_0^1 h(\varphi)d\varphi \le c + C\varphi \le c + \frac{C}{2\varepsilon^2} + \frac{C\varepsilon^2}{2}|\varphi|^2.$$

Thus, by (2.3),

$$\mathcal{V}(\varphi) \geq \frac{1}{2}\|\varphi\|^2 - \frac{C\varepsilon^2}{2\pi^2}\|\varphi\|^2 - c - \frac{C}{2\varepsilon^2}.$$

Choosing $\varepsilon$ such that $4C\varepsilon^2 = 2\pi^2$ and $c$ such that $2\varepsilon^2 c = -C$, we obtain $\mathcal{V}(\varphi) \geq \frac{1}{4}\|\varphi\|^2$, as required for (i), (ii). Also, multiplying (2.12) by $u_t$ and integrating by parts, we find that

$$\frac{d}{dt}\{\mathcal{V}(u(t))\} = -|u_t(t)|^2$$

so that (iii), (iv) follow.

In this article we will start by studying equilibrium points and their neighborhoods in Sections 4 and 5. We will also consider $\omega$-limit sets of sets, leading to the study of *global attractors* — see Section 7. There are currently certain limiting factors in the study of convergence of attractors and, in Section 6, we study an object which contains the global attractor, namely an *inertial manifold.* Because of its stronger attractivity, it is possible to prove stronger results about the affect of perturbation on the inertial manifold than on the global attractor. Section 8 uses our analysis of Section 4 to derive piecewise continuous error bounds for gradient systems. Finally, in Section 9, we briefly describe numerical methods which preserve the dissipative or gradient structure of Defintions 2.13 and 2.15.

### 2.10 Bibliography

The theory of ordinary differential equations in a Hilbert space that we exploit here may be found in Henry [54] and Pazy [81]; the majority of the results in Appendix A are taken from these two sources.

The basic theory and definitions for dynamical systems in finite dimensions is presented very clearly in Bhatia and Szego [11]. Generalizations to partial differential equations include Babin and Vishik [4], Hale [50], Ladyzhenskaya [72] and Temam [92]. See also Chueshov [18] for a review of the subject. General references concerning the numerical analysis of dynamical systems can be found in Beyn [9], Broomhead and Iserles [14], Kloeden and Palmer [69] and Stuart [88].

## 3 Basic approximation of trajectories

### 3.1 Introduction

Section 3.2 contains our basic assumptions about the approximating semigroup. Roughly this states that the error is small in the $C^1$ sense, uniformly on compact time intervals disjoint from the origin and on bounded sets in $V$. In the remaining sections a variety of perturbations to equations of the form (2.9) are described and shown to satisfy the assumptions of Section 3.2.

In Section 3.3 a spectral method is defined for the abstract evolution equation introduced in Section 2.5 and its existence and regularity properties stated. We then derive error estimates for the spectral method. Theorem 3.6 does this by use of a variation of constants approach to the error analysis. (For an introduction to the use of variation of constants, refer back to the **Important remark** following Theorem 2.6). Theorem 3.7 is an extension to obtain $C^1$ error estimates, namely estimates for the difference between the Fréchet derivatives of the true and spectral solution operators with respect to initial data. Such $C^1$ error estimates can be derived for many methods and are particularly useful in the context of dynamical systems. As mentioned they form the basis of our assumptions outlined in Section 3.2. Their importance follows from the fact that $C^1$ closeness implies closeness of the Lipschitz constants of certain nonlinear operators used in the construction of objects of interest in the context of dynamical systems. We emphasize that the spectral method is *not* being recommended for practical computation; it is simply a good example with which to illustrate the abstract approximation theory that follows in remaining sections.

In Section 3.4 we consider the viscous Cahn–Hilliard equation which can be derived from a simplification of the phase-field model of phase transitions and contains the Cahn–Hilliard equation as a singular limit. This singular limit is studied and the assumptions of Section 3.2 shown to hold for the singular perturbation. Section 3.5 contains a discussion of ordinary differential equations.

Those readers not interested in the error analysis for particular perturbations can jump straight to Section 4 after reading Section 3.2. Thus the important point in Section 3 is to understand the basic Assumptions 3.2 which will be used throughout the remainder of the article.

It is our aim in this article to study the effect of various perturbations over long time intervals. In this context it is clear that the error estimates made in Assumptions 3.2 are of no direct use since they contain constants which typically grow with the time interval under consideration. (Indeed the growth is typically exponential). Sections 4–9 deal with a variety of results enabling us to interpret the relationship between the underlying unperturbed dynamical systems and the perturbed dynamical system over long time intervals.

### 3.2 Approximation assumptions

In the rest of the article we will consider a whole class of approximations to the semigroup $S(t)$ given in Assumptions 2.3 yielding approximate semigroups $S^h(t) : V \mapsto V$ satisfying certain natural approximation properties. In some applications, such as time discretization, $t$ may not take on values in the whole of $\mathbb{R}^+$ but in a subset $g(h)\mathbb{N}^+ = \{0, g(h), 2g(h), \ldots\}$. For example, in backward Euler approximation with time step $h$ we will have $g(h) = h$. To enable us to make statements about time-discrete and time-continuous semigroups $S^h(t)$ we use $\mathbf{S}$ to denote $\mathbb{R}^+$ or $g(h)\mathbb{N}^+$ as appropriate.

We denote the Fréchet derivative of $S^h(t)u$ with respect to $u$ evaluated at a point $v \in V$ by $dS^h(v,t)$. Before stating the basic assumptions concerning $S^h(t)$ we make the following definition.

**Definition 3.1** *The approximation error for the semigroup $S^h(t) : V \mapsto V$ as an approximation to the semigroup $S(t) : V \mapsto V$ generated by the partial differential equation (2.9) at a point $u_0 \in V$ is defined by*

$$E(u_0;t) := S(t)u_0 - S^h(t)u_0.$$

*The Fréchet derivative of $E(u_0;t)$ with respect to $u_0 \in V$ evaluated at a point $v \in V$ is denoted by*

$$dE(v;t) := dS(v;t) - dS^h(v;t).$$

Throughout the remainder of this article we make the following assumption concerning the relationship between the semigroup $S(t)$ and its perturbation $S^h(t)$.

**Assumption 3.2** *For all $u \in B(0,R)$ and all $t \in \mathbf{S}$, $t > 0$, there exist constants $C_i = C_i(t,R) < \infty, i = 1,2$, and a function $\kappa : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that the semigroups $S(\bullet)\bullet$ and $S^h(\bullet)\bullet$ satisfy*

$$\|E(u;t)\| \le C_1 h,$$

$$\|dE(u;t)\| \le C_2 \kappa(h),$$

*where $\kappa(h) \to 0$ as $h \to 0_+$.*

We make this assumption throughout the remainder of the paper. We will not state it explicitly in the results. What makes the analysis particularly challenging, in comparison with analogous theories for ordinary differential equations, is that $C_1, C_2$ are not assumed to be bounded as $t \to 0$. Many perturbations, such as those arising from numerical approximation or singular perturbations of the terms in (2.9), have the property that the $C_i$ are unbounded as $t \to 0$, and hence it is important to incorporate it in our assumptions.

In the remainder of this section we detail a variety of situations in which this assumption can be shown to hold. Examples include spectral approximation of (2.9) based on the eigenfunctions of $A$, time approximation of (2.9) based on the backward Euler method and singular perturbations of the Cahn–Hilliard equation arising by considering (2.21) as singular limit of the phase-field equations for phase transitions.

### 3.3 Spectral method for the sectorial equation

We now introduce a spectral method for the approximation of the abstract sectorial evolution equation (2.9) under (2.11). Let $\mathbb{P}$ denote the projection of $X$ into $\text{span}\{\varphi_j\}_{j=1}^N$; thus, given $v$ expressed as

$$v = \sum_{j=1}^{\infty} v_j \varphi_j,$$

we set

$$\mathbb{P}v = \sum_{j=1}^{N} v_j \varphi_j.$$

We also let $\mathbb{Q}$ denote the orthogonal complement of $\mathbb{P}$ so that $\mathbb{Q} = I - \mathbb{P}$. We use the notation $V^{\mathrm{N}} = \mathbb{P}X$. The Galerkin approximation to (2.9) which we consider is to find $u^{\mathrm{N}} \in V^{\mathrm{N}}$ satisfying

$$\frac{du^{\mathrm{N}}}{dt} + Au^{\mathrm{N}} = \mathbb{P}F(u^{\mathrm{N}}), \quad t > 0, \quad u^{\mathrm{N}}(0) = \mathbb{P}u_0^{\mathrm{N}}. \tag{3.1}$$

As for (2.9) it will also be important to consider the linearized problem

$$\frac{dv^{\mathrm{N}}}{dt} + Av^{\mathrm{N}} = \mathbb{P}dF(u^{\mathrm{N}})v^{\mathrm{N}}, t > 0, \quad v^{\mathrm{N}}(0) = \mathbb{P}\xi^{\mathrm{N}}. \tag{3.2}$$

By methods analogous to those used to prove Theorems 2.6, 2.7 we may prove the following two results about (3.1) and (3.2). The proofs are identical after noting that $|\mathbb{P}| = 1$.

**Theorem 3.3** *Let (2.11) hold. Then, for every $u_0^{\mathrm{N}} \in V$ there exists a unique solution $u^{\mathrm{N}}(t)$ of (3.1). Furthermore, there exists $C_1 = C_1(T, R) > 0$ and $C_2 = C_2(\alpha, T, R)$ such that, for all $t \in (0, T)$ and $u_0^{\mathrm{N}} \in B(0, R)$*

$$\begin{aligned} |u^{\mathrm{N}}(t)|_\alpha &\leq \frac{C_1}{t^{\alpha-\beta}}, \forall \alpha \in [\beta, 1]; \\ \left|\frac{du^{\mathrm{N}}}{dt}(t)\right|_\alpha &\leq \frac{C_2}{t^{\alpha-\beta+1}}, \forall \alpha \in [\beta - 1, 1). \end{aligned} \tag{3.3}$$

*Finally, the semigroup $S^{\mathrm{N}}(t)(t) : V \mapsto V$ defined by $S^{\mathrm{N}}(t)(t)u_0^{\mathrm{N}} = u^{\mathrm{N}}(t)$ satisfies Assumptions* 2.3.

**Theorem 3.4** *Let (2.11) hold. Then, for every $\xi^{\mathrm{N}} \in V$ there exists a mild solution $v^{\mathrm{N}}(t)$ of (3.2). Furthermore, there exists $C = C(T, \|u_0^{\mathrm{N}}\|) > 0$ such that, for all $t \in (0, T)$:*

$$\begin{aligned} \|v^{\mathrm{N}}(t)\| &\leq C\|\xi^{\mathrm{N}}\|; \\ |v^{\mathrm{N}}(t)|_\alpha &\leq \frac{C\|\xi^{\mathrm{N}}\|}{(1-\alpha)t^{\alpha-\beta}}, \quad \forall \alpha \in (\beta, 1). \end{aligned} \tag{3.4}$$

*Finally, if* $v^{\mathrm{N}}(t) = dS^{\mathrm{N}}(t)(u_0{}^{\mathrm{N}};t)\xi^{\mathrm{N}}$, *then* $v^{\mathrm{N}}(t)$ *satisfies (3.2).*

Thus the numerical method (3.1) generates a semigroup $S^{\mathrm{N}}(t) : V \to V$ — denoted by $S^h(t)$ in the foregoing theorems — in such a way that the solution $u^{\mathrm{N}}$ of (3.1) is given by

$$u^{\mathrm{N}}(t) = S^{\mathrm{N}}(t)u_0{}^{\mathrm{N}}.$$

Note also that $S^{\mathrm{N}}(t)$ may be viewed as a mapping from $V^{\mathrm{N}}$ to $V^{\mathrm{N}}$ since it is $V^{\mathrm{N}}$ valued for $t > 0$ and $V^{\mathrm{N}} \subset V$. Here the superscript $N$ is used simply to emphasize the dependence of the numerical method on the dimension of the projection $\mathbb{P}$. This semigroup satisfies properties analogous to those for $S(t)$ :

(1) $S^{\mathrm{N}}(0) = \mathbb{P}$, the projection $V \mapsto V^{\mathrm{N}}$;
(2) $S^{\mathrm{N}}(t+s) = S^{\mathrm{N}}(t)S^{\mathrm{N}}(s) \quad \forall t, s \in \mathbb{R}^+$.

(Actually a true semigroup must satisfy $S^{\mathrm{N}}(0) = I$ but having $S^{\mathrm{N}}(0) = \mathbb{P}$ does not affect the analysis given here in any way. We can view $S^{\mathrm{N}}(t)$ as a true semigroup satisfying $S^{\mathrm{N}}(0) = I$ if we restrict its domain to $V^{\mathrm{N}}$.)

We denote the Fréchet derivative of $S^{\mathrm{N}}(t)u_0$ with respect to $u_0 \in V$, evaluated at a point $v \in V$, by $dS^{\mathrm{N}}(v;t)$.

**Example 3.5** Consider $A$ given by Example 2.1 and the heat equation

$$u_t + Au = 0, \quad u(0) = u_0 \in H_0^1(\Omega).$$

For $v \in H_0^1(\Omega) \subset L_2(\Omega)$ we may write $v$ as a series as in (2.2). Similarly any $\xi \in H_0^1(\Omega)$ can be written as

$$\xi = \sum_{j=1}^{\infty} \xi_j \varphi_j.$$

It follows that

$$S(t)v = \sum_{j=1}^{\infty} e^{-\lambda_j t} v_j \varphi_j.$$

Furthemore we have that

$$dS(v;t)\xi = \sum_{j=1}^{\infty} e^{-\lambda_j t} \xi_j \varphi_j.$$

Note that $dS(v;t)$ is independent of $v$ because of the linearity of the problem.

The spectral approximation generates semigroup $S^{\mathrm{N}}(t)$ satisfying

$$S^{\mathrm{N}}(t)v = \sum_{j=1}^{N} e^{-\lambda_j t} v_j \varphi_j$$

and

$$dS^{\mathrm{N}}(v;t)\xi = \sum_{j=1}^{N} e^{-\lambda_j t}\xi_j\varphi_j.$$

We now prove that the approximation error for (3.1) as an approximation of (2.9) is small in both a $C^0$ and a $C^1$ sense. The closeness of the approximation depends on the regularity of the solution being approximated. For many particular equations it is possible to obtain greater regularity than we prove under the assumptions here; this yields stronger approximation results — see Example 3.8.

**Theorem 3.6** **($C^0$ Error Estimates)** *Let (2.11) hold. There exists a constant $C = C(T, R)$ such that the solutions of (2.9) and (3.1) with $u_0 \in B(0, R)$ satisfy*

$$\|u(t) - u^{\mathrm{N}}(t)\| \leq \frac{C}{t^{1-\beta}}[\|\mathbb{P}(u_0 - u_0{}^{\mathrm{N}})\| + \lambda_{N+1}^{\beta-1}] \quad \forall t \in (0, T].$$

**Proof** Let

$$e(t) = \mathbb{P}u(t) - u^{\mathrm{N}}(t), \quad E(t) = u(t) - u^{\mathrm{N}}(t), \quad q(t) = \mathbb{Q}u(t).$$

(Recall that $\mathbb{Q} = I - \mathbb{P}$.) Note that, if

$$u(t) = \sum_{j=1}^{\infty} u_j\varphi_j$$

then

$$\|q(t)\|^2 = \sum_{j=N+1}^{\infty} \lambda_j^{2\beta}u_j^2 \leq \lambda_{N+1}^{2(\beta-1)} \sum_{j=N+1}^{\infty} \lambda_j^2 u_j^2 \leq \lambda_{N+1}^{2(\beta-1)}|u(t)|_1^2.$$

By applying Theorem 2.6 we obtain, for $C_1 = C_1(T, R)$

$$\|q(t)\| \leq \frac{C_1}{(t\lambda_{N+1})^{1-\beta}}. \tag{3.5}$$

Now note that $e(t)$ satisfies the equation

$$e_t + Ae = \mathbb{P}[F(u) - F(u^{\mathrm{N}})].$$

Use of variation of constants gives

$$e(t) = e^{-At}e(0) + \int_0^t e^{-A(t-s)}\mathbb{P}[F(u(s)) - F(u^{\mathrm{N}}(s))]ds.$$

Taking norms, using (2.11), (10.4), $|\mathbb{P}| = 1$ and Lemma 10.6, we obtain

$$\|e(t)\| \leq \|e(0)\| + \int_0^t \frac{K}{(t-s)^{\beta}}\|E(s)\|ds.$$

Hence, since $E(t) = e(t) + q(t)$, we have from (3.5),

$$\|E(t)\| \le \|\mathbb{P}(u_0 - u_0^{\mathrm{N}})\| + \frac{C_1}{(t\lambda_{N+1})^{1-\beta}} + \int_0^t \frac{K}{(t-s)^\beta}\|E(s)\|ds.$$

Thus

$$\|E(t)\| \le \frac{1}{t^{1-\beta}}\left\{T^{1-\beta}\|\mathbb{P}(u_0 - u_0^{\mathrm{N}})\| + C_1\lambda_{N+1}^{\beta-1}\right\} + \int_0^t \frac{K}{(t-s)^\beta}\|E(s)\|ds.$$

Application of the Gronwall Lemma 10.11 gives the desired result for $E(t) = u(t) - u^{\mathrm{N}}(t)$. □

**Important remark** Note that the error estimate blows up as $t \to 0$. To understand this consider the set of functions $u_0 \in B(0, R)$ and their spectral approximations $u_0^{\mathrm{N}}$. Let $u_0 = w^{\mathrm{N}}$ where

$$w^{\mathrm{N}} = \sum_{j=1}^{\infty} w^{\mathrm{N}}{}_j \varphi_j$$

and

$$w^{\mathrm{N}}{}_j = 0, \; j \ne N+1, \quad w^{\mathrm{N}}{}_{N+1} = R/\lambda_j^\beta.$$

Then $\|w^{\mathrm{N}}\|^2 = R^2$ so that $w^{\mathrm{N}} \in B(0, R)$. But $\mathbb{P}w^{\mathrm{N}} = 0$ and so

$$\|w^{\mathrm{N}} - \mathbb{P}w^{\mathrm{N}}\| = R \quad \forall N \ge 0.$$

Hence there are sequences of functions in $B(0, R)$ for which the spectral approximation of the initial data in $X^\beta$ yields a constant error $R$ for each $N \ge 0$. Thus the estimate $\mathcal{O}(\lambda_{N+1}^{\beta-1})$ for the error cannot hold uniformly for all solutions with initial data in $B(0, R)$ as $t \to 0$. This observation is particular to partial differential equations and does not arise in the approximation of ordinary differential equations. It means that certain proofs employed in ordinary differential equations which rely on small time behavior need to be modified. To understand this further the reader should compare the proofs of results in this article with similar results concerning the convergence under perturbation of invariant sets of dynamical systems in ordinary differential equations given in Stuart [88]. Note also that on any compact time interval $[t_1, t_2]$ disjoint from the origin the error estimate is uniform within a ball $B(0, R)$ of initial data. This is not true of the error estimate in Corollary 4.10 for uniform-in-time approximation of trajectories asymptotic to a stable equilibrium point — see the remark following that corollary.

Recall the standard induced operator norm on linear mappings $L \in \mathcal{L}(V, V)$ given in (2.7).

**Theorem 3.7** **($C^1$ Error Estimates)** *Let* 2.11 *hold. There exists a constant* $C = C(T, \|u_0\|)$ *such that, for any* $\alpha \in (\beta, 1)$*, the Fréchet derivative of* *the approximation error generated by solutions of (2.17) and (3.2) satisfies*

$$\|dS(u_0;t) - dS^{\mathrm{N}}(u_0;t)\| \leq \frac{C}{(1-\alpha)(t\lambda_{N+1})^{\alpha-\beta}} \quad \forall t \in (0,T].$$

**Proof** We let

$$d(t) = \mathbb{P}v(t) - v^{\mathrm{N}}(t), \quad D(t) = v(t) - v^{\mathrm{N}}(t), \quad r(t) = \mathbb{Q}v(t)$$

where $v(t)$ and $v^{\mathrm{N}}(t)$ satisfy (2.17), (3.2) with $\xi^{\mathrm{N}} = \xi$ and with $u(t)$ and $u^{\mathrm{N}}(t)$ generated from (2.9), (3.1) with ${u_0}^{\mathrm{N}} = u_0$. Using the regularity established for $v(t)$ in Theorem 2.7, it follows that

$$\|r(t)\| \leq \frac{C_1\|\xi\|}{(1-\alpha)(t\lambda_{N+1})^{\alpha-\beta}}; \tag{3.6}$$

the derivation is similar to that for (3.5). Now $d(t)$ satisfies the equation

$$d_t + Ad = \mathbb{P}[dF(u)v - dF(u^{\mathrm{N}})v] + \mathbb{P}[dF(u^{\mathrm{N}})(v - v^{\mathrm{N}})].$$

Applying the variation of constants formula we obtain

$$\begin{aligned} d(t) = e^{-At}d(0) &+ \int_0^t e^{-A(t-s)}\mathbb{P}[dF(u)v - dF(u^{\mathrm{N}})v]ds \\ &+ \int_0^t e^{-A(t-s)}\mathbb{P}[dF(u^{\mathrm{N}})(v - v^{\mathrm{N}})]ds. \end{aligned}$$

Applying $A^\beta$, taking norms and using (2.11), (10.4) and Lemma 10.6, we find that

$$\|d(t)\| \leq \|d(0)\| + \int_0^t \frac{K\|u(s) - u^{\mathrm{N}}(s)\|\,\|v(s)\|}{(t-s)^\beta}ds + \int_0^t \frac{K\|D(t)\|}{(t-s)^\beta}ds.$$

Since $\xi^{\mathrm{N}} = \xi$ we have $d(0) = 0$. Also, by Theorem 3.6, we have that since ${u_0}^{\mathrm{N}} = u_0$,

$$\|u(t) - u^{\mathrm{N}}(t)\| \leq \frac{C}{(\lambda_{N+1}t)^{1-\beta}}. \tag{3.7}$$

By Theorem 2.7 we have $\|v(t)\| \leq C\|\xi\|$ and hence, putting all this together, we find that

$$\|d(t)\| \leq \int_0^t \frac{K\|\xi\|}{(t-s)^\beta s^{1-\beta}\lambda_{N+1}^{1-\beta}}ds + \int_0^t \frac{K\|D(s)\|}{(t-s)^\beta}ds. \tag{3.8}$$

It may be shown that, if $\beta > 0$, then

$$\int_0^t \frac{ds}{(t-s)^\beta s^{1-\beta}} \leq \frac{1}{\beta(1-\beta)};$$

we return to the case $\beta = 0$ below. Hence, using (3.6) and (3.8), we find that

$$\begin{aligned}\|D(t)\| &\leq \|d(t)\| + \|r(t)\| \\ &\leq \frac{C_2\|\xi\|}{\beta(1-\beta)\lambda_{N+1}^{1-\beta}} + \frac{C_1\|\xi\|}{(1-\alpha)(\lambda_{N+1}t)^{\alpha-\beta}} + \int_0^t \frac{K\|D(t)\|}{(t-s)^\beta}ds.\end{aligned}$$

Hence

$$\|D(t)\| \leq \frac{C_3\|\xi\|}{(1-\alpha)(\lambda_{N+1}t)^{\alpha-\beta}} + \int_0^t \frac{K\|D(t)\|}{(t-s)^\beta}ds.$$

Applying the Gronwall Lemma 10.11 yields

$$\|v(t) - v^{\mathrm{N}}(t)\| \leq \frac{C\|\xi\|}{(1-\alpha)(t\lambda_{N+1})^{\alpha-\beta}}, \quad \forall t \in (0,T].$$

Since $v(t) = dS(u_0;t)\xi$ and $v^{\mathrm{N}}(t) = dS^{\mathrm{N}}(t)(u_0;t)\xi$ the required bound follows from (2.7).

The case $\beta \ll 1$ can be handled similarly by using the (weaker) error bound

$$\|u(t) - u^{\mathrm{N}}(t)\| \leq \frac{C}{(\lambda_{N+1}t)^{\alpha-\beta}}$$

in place of (3.7); this bound may be derived by modifying the proof of Theorem 3.6. □

**Remark** Recall the remark following Theorem 2.7 which indicates why it is not necessary for us to obtain convergence in the $C^1$ norm at the same rate as in the $C^0$ norm. □.

The following example shows, however, that for many particular problems results far stronger than those proved here will hold.

**Example 3.8** For the heat equation Example 3.5 we have, for $\beta = \frac{1}{2}$ so that $\|\bullet\|$ is the norm on $H_0^1(\Omega)$,

$$E(v;t) = \sum_{j=N+1}^{\infty} e^{-\lambda_j t} v_j \varphi_j.$$

Thus

$$\|E(v;t)\|^2 = \sum_{j=N+1}^{\infty} e^{-2\lambda_j t}\lambda_j v_j^2 \leq e^{-2\lambda_{N+1}t}\|v\|^2.$$

This shows the exponential rate of convergence of the spectral method with respect to $N$, for any fixed $t > 0$. The same rate of convergence holds for $\|dE(v;t)\|$. This rate of convergence is a consequence of the high degree of regularity of the solution for each $t > 0$ given by the exponential decay of the Fourier coefficients. In general the addition of nonlinear terms will destroy this smoothness and hence the weaker error bounds proved in Theorems 3.6, 3.7 are typical. However, many nonlinear problems do possess a form of regularity known as *Gevrey class* which implies rapid decay of the co-efficients in eigenfunction expansions; such regularity can be exploited in the error estimates for spectral methods. Ferrari and Titi [35] give general conditions under which (2.9) yields solutions of Gevrey class regularity.

Note that Theorems 3.6 and 3.7 show that, at any positive finite time, both the semigroup and its derivative are well-approximated. The constants appearing in the error bounds depend only on the norm of the initial data $u_0$ and the time interval under consideration. Thus with the definition

$$S^h(t) := S^{\mathrm{N}}(t) \quad \forall h \in [\lambda_{N+1}^{(1-\beta)}, \lambda_N^{(1-\beta)}), \tag{3.9}$$

Theorems 3.6, 3.7 show that Assumptions 3.2 are satisfied for the spectral approximation.

### 3.4 Backward Euler method for sectorial equations

As another example of a method satisfying the approximation Assumptions 3.2, consider the backward Euler method

$$U^{n+1} - U^n + \Delta t A U^{n+1} = \Delta t F(U^{n+1}), \quad U^0 = u_0.$$

This equation generates an approximation $U^n \approx u(n\Delta t)$. The approximation assumptions used in this paper are established for the method applied to a reaction-diffusion equation and to the Kuramtoto-Sivashinsky equation respectively in Hale *et al.* [51] and Alouges and Debussche [1].

### 3.5 The phase-field and viscous Cahn–Hilliard equations

Now we introduce an example where the effect of perturbation is not from numerical approximation but from a singular perturbation to the partial differential equation. The phase-field equations are

$$c\theta_t + \frac{l}{2}u_t = k\Delta\theta, \quad x \in \Omega, \quad t > 0,$$
$$\alpha u_t = \Delta u + f(u) + \delta\theta, \quad x \in \Omega, \quad t > 0.$$

We consider these equations subject to the Dirichlet boundary conditions

$$u = \theta = 0 \qquad x \in \partial\Omega, \quad t > 0,$$

and initial conditions on $u$ and $\theta$. These equations model phase transitions such as that between ice and water; see Caginalp [15].

If we set $c = 0$, re-scale time and the function $f$ and introduce a scaled version of $\theta$, namely $w$, then we obtain equations in the form

$$\begin{aligned} u_t &= \Delta w, \quad x \in \Omega, \\ \epsilon u_t &= \Delta u + f(u) + w, \quad x \in \Omega, \\ u &= w = 0, \quad x \in \partial\Omega, \\ u &= u_0(x), \quad t = 0. \end{aligned} \tag{3.10}$$

These are a form of the viscous Cahn–Hilliard equation. We use $A_0$ to denote the same operator as in (2.21) and then (3.10) may be written as

$$u_t + (\varepsilon + A_0^{-1})^{-1} A_0 u = (\varepsilon + A_0^{-1})^{-1} F(u).$$

If $\epsilon$ is small this may be viewed as a non-local singular perturbation of the Cahn–Hilliard equation (2.21). In Elliott and Stuart [32] it is shown that for any $\varepsilon \geq 0$ equations (3.10) generate a dynamical system with semigroup $S(\bullet)\bullet \in C^2(\mathbb{R}^+ \times V, V)$ where $V = H_0^1(\Omega)$ so that Assumptions 2.3 hold. Furthermore the semigroup is shown to be $C^1$ in $\epsilon$, uniformly on compact time intervals disjoint from the origin and on bounded sets in $V$. Thus Assumptions 3.2 hold with $h = \kappa(h) = \epsilon$.

### 3.6 Ordinary differential equations

The system of ordinary differential equations

$$u_t = f(u; \epsilon), \quad u(0) = U,$$

where $f$ is smooth in $u$ and $\epsilon$, generates a semigroup satisfying Assumptions 2.3 and 3.2 with $h = \kappa(h) = \varepsilon$. Thus the theory in this article applies.

### 3.7 Bibliography

The approach to $C^0$ error estimates given here is similar to that developed for finite element methods applied to reaction-diffusion equations in Larsson [74] and generalized to the Cahn–Hilliard equation in Elliott and Larsson [30]; that work in turn builds on the work of Thomee [96] and of Johnson *et al.* [61]. These works employ the semigroup approach to facilitate the error analysis. The use of $C^1$ error estimates is particularly important in the context of dynamical systems and examples of such results may be found in Alouges and Debussche [1] and Hale *et al.* [51] for specific approximations of reaction-diffusion equations, in Stuart [88] for arbitrary one-step methods applied to ordinary differential equations and in Jones and Titi [64] and Jones [62] for spectral and finite difference approximations of certain partial differential equations of the form (2.9).

The spectral method based on the eigenfunctions of $A$ is typically only useful as a numerical computational technique for problems with simple geometry and simple differential operators such as those appearing in Examples 2.1, 2.5. An early use of such a computational method for (2.9) is described in Orszag [80]; a more recent reference, directly relevant to the computation of invariant manifolds in partial differential equations, is Bai *et al.* [5]. The spectral method (3.1) is also of theoretical importance as a tool to prove existence results about (2.9); indeed, by use of a spectral method, such results can be proved under weaker hypotheses concerning the initial data $u_0$ than those given here – see Friedman [40], Lions [76], Constantin and Foias [19], and Temam [91] [92].

## 4 Equilibria and phase portraits

### 4.1 Introduction

This section starts with a consideration of the effect of the approximation error on general, not necessarily stable, equilibrium points. The approach is to use the implicit function theorem. The core of the analysis is contained in Lemma 4.2 but the main result is stated as Theorem 4.3.

In Section 4.3 we study the one case where error estimates for (2.9) may be obtained which are independent of the time-interval, namely for solutions approaching an exponentially stable equilibrium point. This introduces the important idea, which we use throughout the article, of combining the standard finite time error estimates together with some underlying property concerning the behavior of the differential equation. The most important result in this context is Corollary 4.10. An important steppingstone along the way is Lemma 4.7 which shows that, near a stable hyperbolic equilibrium point, the semigroup is contractive.

In Section 4.4 we move from stable equilibria to local phase portraits near saddle points — the union of all solutions of (2.9) in a small ball around the equilibrium point. Since such solutions can spend an arbitrarily long time near the equilibrium point standard error analysis does not apply. The key is to compare true and numerical solutions with different initial data and the important results are Theorems 4.18 and 4.19. The method of analysis used in this section is to construct the solutions of interest by use of a contraction mapping argument (see Theorem 4.13) and then use the uniform contraction principle (see Appendix B) to incorporate the effect of perturbation.

### 4.2 Equilibria

In this section we are concerned with the behavior of steady solutions of (2.9) and solutions in their neighborhood.

**Definition 4.1** *A point $\bar{u} \in V$ is a fixed point of $S(\tau)$ for some $\tau \in \mathbb{R}$ if $S(\tau)\bar{u} = \bar{u}$. A point $\bar{u} \in V$ is an equilibrium point if it is a fixed point of $S(t)$ for all $t \in \mathbb{R}$. A fixed point $\bar{u}$ of a semigroup $S(t)$ is said to be hyperbolic if $dS(\bar{u}, t)$ has no eigenvalues on the unit circle. An equilibrium point $\bar{u}$ of a semigroup $S(t)$ is said to be hyperbolic if $dS(\bar{u}, t)$ has no eigenvalues on* *the unit circle for all* $t \neq 0$. □

Throughout we will use the following notation for the set of fixed points of $S(t)$ and $S^h(t)$; recall the notation $\mathbf{S}$, given before Definition 3.1, enabling us to consider time-continuous and time-discrete semigroups together.

$$\begin{aligned} \mathcal{E} &= \{v \in V : S(t)v = v \ \ \forall t \in \mathbb{R}\} \\ \mathcal{E}^h &= \{v \in V : S^h(t)v = v \ \ \forall t \in \mathbf{S}\} \end{aligned} \tag{4.1}$$

Let us assume that $S(t)$ has an equilibrium point $\bar{u}$. We introduce the new variable

$$v(t) = u(t) - \bar{u}$$

and change variables in (2.9). Then $v(t)$ satisfies the equation

$$\begin{aligned} &v_t + Cv = g(v), \quad v(0) = v_0 := u_0 - \bar{u}, \\ &C = A - dF(\bar{u}), \quad g(v) = [F(v+\bar{u}) - F(\bar{u}) - dF(\bar{u})v]. \end{aligned} \tag{4.2}$$

Recall the operator norm of $L \in \mathcal{L}(X,X)$ given by (2.8). Note that $|(C-A)A^{-\beta}| = |dF(\bar{u})A^{-\beta}| \leq K$ by (2.11) and hence, by Lemma 10.13, the operator $C$ is sectorial so that $e^{-Ct}$ may be defined. Thus we see that $dS(\bar{u},t) = e^{-Ct}$.

The hyperbolicity of $\bar{u}$ is equivalent to the operator $C$ having no eigenvalues with zero real part. When $\bar{u}$ is hyperbolic, by Theorem 10.14 we can split the space $X$ into $X = Y' \oplus Z'$ where $Y'$ (resp. $Z'$) is the subspace of $X$ spanned by the generalized eigenspace of $C$ corresponding to eigenvalues with negative (resp. positive) real parts. We denote by $\mathcal{P}$ and $\mathcal{Q}$ the spectral projections $\mathcal{P} : X \to Y'$ and $\mathcal{Q} : X \to Z'$ and then denote $Y = \mathcal{P}V, Z = \mathcal{Q}V$ so that $V = Y \oplus Z$. Using Theorem 10.15 it follows that, for any $a < 1$, there exists $T^* > 0$ such that

$$\begin{aligned} &\|e^{Ct}v\| \leq a\|v\| \quad \forall t \geq T^*, \ \forall v \in Y \\ &\|e^{-Ct}v\| \leq a\|v\| \quad \forall t \geq T^*, \ \forall v \in Z. \end{aligned} \tag{4.3}$$

Our aim is to show that if $\bar{u}$ is hyperbolic then, under Assumptions 3.2, $S^h(t)$ will also have a fixed point and hence an equilibrium point. We start by showing in the next lemma that for any given fixed time $t$ the semigroup $S^h(t)$ has a fixed point $\bar{u}^h(t)$. In the theorem following the lemma we establish that the fixed point of $S^h(t)$ is actually an equilibrium point for $S^h(t)$.

**Lemma 4.2** *Assume that* $S(\bullet)\bullet \in C^2(\mathbb{R}^+ \times V, V)$. *Let* $\bar{u}$ *be a hyperbolic equilibrium point of (2.9) and assume that* $t \geq T^*$ *given by (4.3) and that* $t \in \mathbf{S}$. *Then there exists* $h^*, C > 0$ *such that* $S^h(t)$ *has a fixed point,* $\bar{u}^h = \bar{u}^h(t)$ *unique in* $B(\bar{u}; Ch)$, *for all* $h \in (0, h^*]$.

**Proof** The proof is similar to the proof of the implicit function theorem. Consider the mapping

$$\begin{aligned} W^{k+1} &= F(W^k;t) \\ F(W;t) &:= W - D[W - S^h(t)W] \end{aligned} \tag{4.4}$$

where $D = D(t) = [I - dS(\bar{u},t)]^{-1}$ and $t \geq T^*$, where $T^*$ is given in (4.3). As a preliminary to the proof we prove that $\|D\|$ is bounded. Consider the equation

$$(I - e^{-Ct})\mathbf{a} = \mathbf{b}$$

and note that, in the induced operator norm (2.7) we have

$$\|D\| = \sup_{\|\mathbf{b}\|=1} \|\mathbf{a}\|,$$

since $dS(\bar{u},t) = e^{-Ct}$. We write $\mathbf{a} = a_p + a_q$ and $\mathbf{b} = b_p + b_q$ where $a_p, b_p \in Y$ and $a_q, b_q \in Z$; here $Y$ and $Z$ are the invariant subspaces of $C$. Thus

$$\begin{aligned} (I - e^{-Ct})a_p &= b_p \\ (I - e^{-Ct})a_q &= b_q. \end{aligned}$$

Using the hyperbolicity property we deduce that (4.3) holds and hence that

$$\begin{aligned} \|a_p\| &\leq a(1-a)^{-1}\|b_p\| \\ \|a_q\| &\leq (1-a)^{-1}\|b_q\|. \end{aligned} \tag{4.5}$$

Note that $\| \bullet \|$ is equivalent to the norm $\| \bullet \|_V$ given by

$$\| \bullet \|_V = \max\{\|\mathcal{P} \bullet\|, \|\mathcal{Q} \bullet\|\}.$$

Hence it follows from (4.5) that there exists a constant $K > 0$ such that

$$\|\mathbf{a}\| \leq K(1-a)^{-1}\|\mathbf{b}\|$$

so that $\|D\| \leq K(1-a)^{-1}$.

Fix $\varepsilon > 0$. Recall the constants $C_1 = C_1(t, \|\bar{u}\| + \varepsilon)$ and $C_2 = C_2(t, \|\bar{u}\| + \varepsilon)$ given by Assumptions 3.2 and evaluated here at time $t$ and within a ball of radius $\|\bar{u}\| + \epsilon$. To prove existence of a fixed point of (4.4) we show that the iteration (4.4) maps $B(\bar{u}; Ch)$ into itself for $C = (1+\alpha)\|D\|C_1$ and that it is a contraction on that set. The constant $\alpha$ has been introduced to facilitate the proof of Theorem 4.3 following this lemma. We assume that $h$ is sufficiently small that

$$(1+\alpha)\|D\|C_1 h \leq \varepsilon \tag{4.6}$$

so that $u \in B(\bar{u}; Ch)$ implies that $\|u\| \leq \|\bar{u}\| + \varepsilon$; thus, since all our analysis takes place in $B(\bar{u}; Ch)$, the constants $C_1$ and $C_2$ as defined are the appropriate constants in the following argument. Clearly a fixed point of the mapping (4.4) is necessarily a fixed point of $S^h(t)$. To show that the mapping (4.4) is into, note that by Definition 3.1, (4.4) may be written as

$$W^{k+1} = W^k - D[W^k - S(t)W^k + E(W^k;t)]. \tag{4.7}$$

Also, since $\bar{u}$ is a fixed point of $S(t)$ it follows that

$$\bar{u} = \bar{u} - D[\bar{u} - S(t)\bar{u}]. \tag{4.8}$$

Let $W^k \in B(\bar{u}; Ch)$ and set $e^k = W^k - \bar{u}$. Then (4.7), (4.8) yield, upon appplication of Taylor's Theorem 11.4,

$$\|e^{k+1}\| = \|e^k - D[(I - dS(\bar{u},t))e^k + Q_1 + E(W^k;t)]\|$$

where

$$\|Q_1\| \leq K_1\|e^k\|^2,$$

for some $K_1 > 0$. Thus, using the definition of $D$,

$$\|e^{k+1}\| \leq \|D\|\|Q_1\| + \|D\|\|E(W^k;t)\|.$$

Hence, by Assumption 3.2,

$$\|e^{k+1}\| \leq (1+\alpha)^2\|D\|^3K_1C_1^2h^2 + \|D\|C_1h.$$

Choosing $h$ sufficiently small so that

$$(1+\alpha)^2K_1\|D\|^2C_1h \leq \alpha, \tag{4.9}$$

we deduce that the mapping (4.4) takes $B(\bar{u}, Ch)$ into itself.

To show that the mapping (4.4) is a contraction, let $V^k$ satisfy (4.7) with $W^k \to V^k$ and define $d^k = W^k - V^k$; assume that $W^0, V^0 \in B(\bar{u}, Ch)$. Then

$$\|d^k\| \leq 2(1+\alpha)\|D\|C_1h. \tag{4.10}$$

A similar manipulation to that used in showing that the mapping (4.4) is "into" yields

$$\|d^{k+1}\| \leq \|d^k - D[(I - dS(\bar{u},t))d^k + Q_2 + E(W^k;t) - E(V^k;t)]\|$$

where

$$\|Q_2\| \leq K_2\|d^k\|^2,$$

for some constant $K_2 > 0$. Hence, by Assumption 3.2 and Taylor's Theorem 11.4, (4.10) gives

$$\begin{aligned} \|d^{k+1}\| &\leq \|D\|\|Q_2\| + \|D\|\|E(W^k;t) - E(V^k;t)\| \\ &\leq 2(1+\alpha)\|D\|^2K_2C_1h\|d_k\| + \|D\|C_2\kappa(h)\|d_k\|. \end{aligned} \tag{4.11}$$

Thus the mapping (4.4) is a contraction for $h$ sufficiently small so that

$$2(1+\alpha)K_2\|D\|^2C_1h + \|D\|C_2\kappa(h) \le \frac{1}{2}. \tag{4.12}$$

The existence of a fixed point $\bar{u}^h$ of $S^h$ follows for $h \le h^* = h^*(\|\bar{u}\|, \varepsilon, t, \alpha)$ (given by (4.6), (4.9), (4.12) and $t \ge T^*$.) □

Now we deduce that the fixed point $\bar{u}^h(t)$ from the previous lemma is actually independent of $t$ and hence an equilibrium point of $S^h(t)$.

**Theorem 4.3 (Equilibrium Points Under Approximation)** *Assume that $S(\bullet)\bullet \in C^2(\mathbb{R}^+ \times V, V)$ and assume that $S^h(t)$ is a time-continuous semigroup. Let $\bar{u}$ be a hyperbolic equilibrium point of (2.9). Then there exist $h_c > 0$ and $C > 0$ such that $S^h(t)$ has an equilibrium point $\bar{u}^h$, unique in $B(\bar{u}; Ch)$, for all $h \in (0, h_c]$.*

**Proof** Consider the fixed point $\bar{u}^h$ found in Lemma 4.2 by setting $t = T \ge T^*$. First we show that the point $\bar{u}^h$ is a fixed point of $S^h(t)$ for all $t$. Note that $h^*$ and $C$ in the previous theorem both depend upon $\alpha$; since this is the only dependency of interest to us here we denote these quantities by $h^*(\alpha)$ and $C(\alpha)$; recall that $\alpha$ was introduced to parameterize the radius of the ball in which the contraction argument of Lemma 4.2 was performed and note that $C(1) < C(2)$. By Lemma 4.2 we deduce that, if $h \le \max\{h^*(1), h^*(2)\}$, then there is a fixed point of $S^h(t)$ in $B(\bar{u}, C(1)h)$ which is unique in $B(\bar{u}, C(2)h)$. Clearly $\bar{u}^h$ must lie on a periodic solution with minimum period $\tau$ where, if $\tau \ne 0$, then $T$ is an integer multiple of $\tau$, say $T = m\tau$. We wish to show that $\tau = 0$. Denote the set of points on the periodic solution by

$$\mathcal{S} = \{S^h(t)\bar{u}^h : 0 \le t \le \tau\}.$$

If $\tau > 0$ then, by continuity, there exists $v \in \mathcal{S}$ with $v \ne \bar{u}^h$ and $v \in B(\bar{u}, C(2)h)$ and satisfying $S^h(m\tau)v = v$. This contradicts the uniqueness of $\bar{u}^h$ in $B(\bar{u}, C(2)h)$ and it follows that $\tau = 0$ as required. Thus $\bar{u}^h(t)$ is actually an equilibrium point of $S^h(t)$.

To see that this equilibrium point is unique in a ball of sufficiently small radius note that, if it is not, then Lemma 4.2 is contradicted since any other equilibrium point would also be a fixed point of $S^h(t)$. □

A similar argument proves the following, an analog of Theorem 4.3 when $S^h(t)$ is time-discrete.

**Theorem 4.4 (Equilibrium Points Under Approximation)** *Assume that $S(\bullet)\bullet \in C^2(\mathbb{R}^+ \times V, V)$ and assume that $S^h(t)$ is a time-discrete semigroup so that $S^h(g(h))$ is a one-step map from $V$ into $V$. Let $\bar{u}$ be a hyperbolic equilibrium point of (2.9). Then there exists $h_c, C > 0$ such that $S^h(g(h))$ has a fixed point $\bar{u}^h$, unique in $B(\bar{u}; Ch)$, for all $h \in (0, h_c]$.*

### 4.3 Trajectories asymptotic to a stable steady state

We continue our analysis of the approximation of dynamical systems by generalizing Example 3.8, which shows uniform convergence for $t \in (0, \infty)$ for a solution approaching an exponentially stable equilibrium point of the linear heat equation, to the general nonlinear problem. In the following section we consider the neighborhood of an equilibrium of saddle type and derive uniform-in-time approximation results for trajectories near the saddle. In Section 5 we study unstable manifolds near a saddle point. Thus for both this section, the following section and Section 5, we assume that there exists $\bar{u} \in D(A)$ such that

$$A\bar{u} = F(\bar{u}). \tag{4.13}$$

We introduce the notation $\bar{S} : V \to V$ to denote the semigroup constructed so that $v(t) = \bar{S}(t)v_0$ solves (4.2). Hence

$$\bar{S}(t)v = S(t)(\bar{u} + v) - \bar{u}. \tag{4.14}$$

For partial differential equations of the form (2.9) it follows from (4.2) that the nonlinear function $g(\bullet)$ is quadratic in $v$. Thus we make the following assumption:

**Assumption 4.5** *There exists $\zeta < 1 - \beta$ such that*

$$\|g(v) - g(w)\|_{-\zeta} \le k(\rho)\|v - w\|_{\beta} \quad \forall v, w \in B(0, \rho). \tag{4.15}$$

*Here $k : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is nondecreasing and satisfies*

$$k(\rho) \to 0 \quad as \quad \rho \to 0_+. \tag{4.16}$$

The following example illustrates the assumption:

**Example 4.6** Consider the reaction-diffusion equation

$$u_t = \Delta u + f(u), \quad x \in \Omega$$

$$u = 0, \quad x \in \partial\Omega$$

$$u = u_0, \quad t = 0.$$

If $\Omega$ is a sufficiently smooth domain in $\mathbb{R}^d$ and $f(u)$ satisfies

$$|f^{(j)}(u)| \le C(1 + |u|^{\delta - j}), \quad j = 0, 1, 2, \ u \in \mathbb{R},$$

for some $C > 0$ where $\delta = 3$ if $d = 3$ and $\delta < \infty$ if $d < 3$, then (4.15) and (4.16) hold for $g$ given by (4.2). In that case $\beta = \frac{1}{2}$ and $\zeta = 0$. See Larsson and Sanz-Serna [75] for details.

Consider the Cahn–Hilliard equation (2.21) which can be written as

$$u_t = -\Delta\{\Delta u + f(u)\}, \quad x \in \Omega$$

$$u = \Delta u + f(u) = 0, \quad x \in \partial\Omega$$

$$u = u_0, \quad t = 0.$$

If $f$ satisfies the conditions described for the preceding reaction-diffusion equation then (4.15), (4.16) hold with $\beta = \frac{1}{4}$ and $\zeta = \frac{1}{2}$. See Elliott and Stuart [32] for details.

This assumption about $g$ enables us to prove the following result about the semigroup in the neighborhood of stable equilibria. Recall Definition 2.12.

**Lemma 4.7** *Let Assumption* 4.5 *hold. If the equilibrium point $\bar{u}$ is hyperbolic and stable (so that $Z \equiv V$ and $Y \equiv \emptyset$) then the semigroup generated by (2.9) is contractive in a neighborhood of $\bar{u}$.*

**Proof** We work in the $v$ variable so that $\bar{u}$ translates to the origin. We may use the variation of constants formula to write the solution of (4.2) as

$$v(t) = e^{-Ct}v(0) + \int_0^t e^{-C(t-s)}g(v(s))ds. \tag{4.17}$$

Thus a second solution $w(t)$ satisfies

$$w(t) = e^{-Ct}w(0) + \int_0^t e^{-C(t-s)}g(w(s))ds.$$

Letting $e(t) = v(t) - w(t)$, applying Theorem 10.15 and using Assumption 4.5 we find that

$$\|e(t)\| \leq C_1 e^{-\gamma t}\|e(0)\| + \int_0^t \frac{k(\rho)C_1 e^{-\gamma(t-s)}\|e(s)\|}{(t-s)^{\beta+\zeta}}ds, \tag{4.18}$$

provided $v(t)$, $w(t) \in B(0,\rho)$.

By Lemma 10.12 there exists $K > 0$ such that, for $\nu = 1 - \beta - \zeta > 0$,

$$\|e(t)\| \leq 2C_1\|e(0)\| \exp[\{K(C_1 k(\rho))^{1/\nu} - \gamma\}t]$$

whilst $v(t), w(t) \in B(0,\rho)$. Choose $\rho$ sufficiently small that

$$K(C_1 k(\rho))^{1/\nu} \leq \frac{\gamma}{2} \tag{4.19}$$

and let $v(0) = 0$ so that $v(t) = 0$ for all $t \geq 0$. If $\|w(0)\| \leq \sigma := \rho/2C_1$ then, arguing by contradiction, we have that $w(t) \in B(0,\rho)$ for all $t \geq 0$

and, furthermore, that

$$\|w(t)\| \le \rho e^{-\gamma t/2} \quad \forall t \ge 0. \tag{4.20}$$

Hence any two solutions with $v(0), w(0) \in B(0,\sigma)$ satisfy

$$\|v(t) - w(t)\| \le 2C_1 e^{-\gamma t/2}\|v(0) - w(0)\|.$$

If $\tau = 4\log(2C_1)/\gamma$ then

$$\|v(t) - w(t)\| \le e^{-\gamma t/4}\|v(0) - w(0)\| \quad \forall t \ge \tau. \tag{4.21}$$

Thus, by Definition 2.12, contractivity holds at the origin. Converting back to the $u$ variable gives the desired result with

$$\alpha = \gamma/4, \quad \tau = 4\log(2C_1)/\gamma, \quad \sigma = \rho/2C_1 \tag{4.22}$$

and $\rho$ sufficiently small that (4.19) holds. □

**Theorem 4.8 (Time Uniform Approximation of Trajectories)** *Let Assumption 4.5 hold and let $\bar{u}$ be hyperbolic and stable. Then there exists $\alpha > 0$ such that, for any $u_0 \in B(0,\sigma)$, the solution of (2.9) satisfies*

$$\|S(t)u_0 - \bar{u}\| \le e^{-\alpha t}\|u_0 - \bar{u}\| \quad \forall t \ge \tau.$$

*Furthermore, there exists a constant $K = K(\tau, \|\bar{u}\|, \sigma) > 0$ such that, for any $u_0 \in B(0,\sigma/2)$ the semigroup $S(t)$ generated by the solution of (2.9) and the approximate semigroup $S^h(t)$ satisfy*

$$\|S(t)u_0 - S^h(t)u_0\| \le Kh \quad \forall t \ge 2\tau,$$

*if $h$ is sufficiently small.*

**Proof** The first result follows by taking $u_1 = u_0, u_2 = \bar{u}$ in Definition 2.12 and noting that $S(t)\bar{u} = \bar{u}\ \forall t \ge 0$. Note that, if $\xi \ge 1$ and $u_0 \in B(0,\sigma/\xi)$, then $S(t)u_0 \in B(0,\sigma/\alpha)$ for all $t \ge \tau$ by (4.21). Let $r = \|\bar{u}\| + \sigma$ and note that

$$\|S(t)u - S^h(t)v\| \le C(t,r)[\|u - v\| + h] \quad \forall u, v \in B(0,\sigma), \tag{4.23}$$

where we may assume that $C(t,r)$ is non-decreasing in $t$ for $t \ge \tau$ without loss of generality — this follows from Assumption 3.2 and Lemma 2.4. Also $C(t,r) < \infty$ for any $t \in (0,\infty)$. Let

$$E(t) = \|S(t)u_0 - S^h(t)u_0\|, \quad E_m := E(m\tau).$$

Assume for induction that

$$\|E_m\| \le \left[\frac{1 - e^{-m\alpha\tau}}{1 - e^{-\alpha\tau}}\right] C(\tau,r)h, \tag{4.24}$$

noting that this holds for $m = 1$ by (4.23) with $u = v = u_0$ and $t = \tau$. Assume that $h$ is sufficiently small so that

$$\|E_m\| \leq \frac{C(\tau, r)h}{1 - e^{-\alpha\tau}} \leq \sigma/2.$$

Since $S(m\tau)u_0 \in B(0, \sigma/2)$ it follows that $S^h(m\tau)u_0 \in B(0, \sigma)$. Now, by Definition 2.12, (4.21), (4.22) and (4.23),

$$\begin{aligned}
\|E_{m+1}\| &= \|S(\tau)S(m\tau)u_0 - S^h(\tau)S^h(m\tau)u_0\| \\
&\leq \|S(\tau)S(m\tau)u_0 - S(\tau)S^h(m\tau)u_0\| \\
&\qquad + \|S(\tau)S^h(m\tau)u_0 - S^h(\tau)S^h(m\tau)u_0\| \\
&\leq e^{-\alpha\tau}\|E_m\| + C(\tau, r)h.
\end{aligned}$$

Hence, by (4.24), we have

$$\begin{aligned}
\|E_{m+1}\| &= e^{-\alpha\tau}\left(\frac{1 - e^{-m\alpha\tau}}{1 - e^{-\alpha\tau}}\right) C(\tau, r)h + C(\tau, r)h \\
&\leq \left(\frac{1 - e^{-(m+1)\alpha\tau}}{1 - e^{-\alpha\tau}}\right) C(\tau, r)h.
\end{aligned}$$

Thus (4.24) holds for all $m \geq 1$ by induction so that

$$\|E_m\| \leq \frac{C(\tau, r)h}{1 - e^{-\alpha\tau}} \quad \forall m \geq 1.$$

It remains to fill in the error between times $t = m\tau$ for $m \geq 2$. By (4.23), if $t = m\tau + T$ with $m \geq 1$ and $T \in [\tau, 2\tau)$, then

$$\begin{aligned}
\|S(t)u_0 - S^h(t)u_0\| &\leq \|S(T)S(m\tau)u_0 - S^h(T)S^h(mT)u_0\| \\
&\leq C(T, r)[\|E_m\| + h] \\
&\leq C(2\tau, r)[\|E_m\| + h].
\end{aligned}$$

The result follows. □

The following is a straightforward corollary of Theorem 4.8.

**Corollary 4.9 (Time Uniform Approximation of Trajectories)** *Let Assumption* 4.5 *hold and let* $\bar{u}$ *be a hyperbolic, stable equilibrium point. Then, for any* $u_0, u_0^h \in B(0, \sigma/2)$*, there exists* $K = K(\tau, \|\bar{u}\|, \sigma)$ *such that*

$$\|S(t)u_0 - S^h(t)u_0^h\| \leq e^{-\alpha t}\|u_0 - u_0^h\| + Kh \quad \forall t \geq 2\tau,$$

*for all* $h$ *sufficiently small.*

**Proof** Note that, by Theorem 4.8 and Definition 2.12,

$$\begin{aligned}\|S(t)u_0 - S^h(t)u_0^{\mathrm{h}}\| &\le \|S(t)u_0 - S(t)u_0^{\mathrm{h}}\| + \|S(t)u_0^{\mathrm{h}} - S^h(t)u_0^{\mathrm{h}}\| \\ &\le e^{-\alpha t}\|u_0 - u_0^{\mathrm{h}}\| + Kh\end{aligned}$$

for all $t \ge 2\tau$. □

**Corollary 4.10 (Time Uniform Approximation of Trajectories)** *Let Assumption 4.5 hold and let $\bar{u}$ be hyperbolic and stable. Assume also that the solution of (2.9) satisfies $u(t) \to \bar{u}$ as $t \to \infty$. Then there exists a constant $K_1 = K_1(\tau, \|\bar{u}\|, u_0) > 0$ such that the semigroup $S(t)$ generated by the solution of (2.9) and the approximate semigroup $S^h(t)$ satisfy*

$$\|S(t)u_0 - S^h(t)u_0\| \le K_1 h \quad \forall t \ge 2\tau,$$

*for all $h$ sufficiently small.*

**Proof** If $u_0 \in B(0, \sigma/2)$ the result follows directly from Theorem 4.8. If $u_0 \notin B(0,\sigma/2)$ then there exists $T = T(u_0)$ such that $S(t)u_0 \in B(0, \sigma/4)$ for all $t \ge T$, since $u(t) \to \bar{u}$. Without loss of generality we may assume that $T \ge 2\tau$. By Assumption 3.2 there is $C = C(T + 2\tau, \|u_0\|)$ such that

$$\|S(t)u_0 - S^h(t)u_0\| \le Ch \quad \forall t \in [2\tau, T + 2\tau].$$

By choosing $h$ sufficiently small we have $S^h(t)u_0 \in B(0,\sigma/2)$. Applying Corollary 4.9 gives $K = K(\tau, \|\bar{u}\|, \sigma)$ such that

$$\begin{aligned}\|S(t)u_0 - S^h(t)u_0\| &= \|S(t-T)S(T)u_0 - S^h(t-T)S^h(T)u_0\| \\ &\le e^{-\alpha(t-T)}\|S(T)u_0 - S^h(t)u_0\| + Kh \\ &\le (C+K)h,\end{aligned}$$

for all $t \ge T + 2\tau$. This completes the proof by defining $K_1 = C + K$. □

**Important remark** Note that $K$, the error constant, depends explicitly on $u_0$. Furthermore it is not necessarily uniform in $u_0 \in B(0,\rho)$. This is since it is possible for there to be a sequence ${u_0}^{(i)} \to {u_0}^{(\infty)}$ for which $S(t){u_0}^{(i)} \to \bar{u}$ for each $i$ but $S(t){u_0}^{(\infty)}$ does not tend to $\bar{u}$. By continuity it follows that $T({u_0}^{(i)}) \to \infty$ as $i \to \infty$ and hence $K(\tau, \|\bar{u}\|, u_0)$ is not uniformly bounded.

This lack of uniformity is undesirable in some circumstances since $T(u_0)$ is usually not known a priori. By weakening the notion of approximation to piecewise approximation, the non-uniformity with respect to initial data can be overcome in some circumstances. See Section 8. □

## 4.4 Phase portraits near a saddle

In this section our aim is to construct solutions of (2.9) (or equivalently (4.2)) in the neighborhood of a hyperbolic equilibrium point $\bar{u} \in \mathcal{E}$ (or equivalently $v = 0$) of saddle type. The union of these solutions will form a "local phase portrait" near $\bar{u}$. We will then show that, under Assumptions 3.2, each of these solutions can be approximated by $S^h(t)$ *independently of the time interval* over which the solution is being considered; thus the complete local phase portrait perturbs smoothly with $h$. Recall that standard error estimates involve constants which grow exponentially with the time interval under consideration.

By virtue of the smallness properties of $g(\bullet)$ given by Assumptions 4.5 it is reasonable to expect that, for hyperbolic equilibria, the properties of the linear equation $w_t + Cw = 0$ describe the dynamics of solutions to (4.2) in the neighborhood of $v = 0$. Our aim is to prove such a result and then show continuity with respect to the perturbations introduced by the approximating semigroup of Assumption 3.2.

We now formulate the solution of (4.2) as a mapping over time interval $T$. Recalling (4.14), (4.17) we may write

$$v(t) = L(t)v(0) + G(v(0), t) \tag{4.25}$$

where

$$L(t) := e^{-Ct}, \quad G(v,t) := \int_0^t L(t-s)g(\bar{S}(s)v)ds. \tag{4.26}$$

Now we define $t_n = nT$, $v_n = v(t_n)$ and deduce from (4.25) that

$$v_{n+1} = Lv_n + G(v_n) \tag{4.27}$$

where $L := L(T)$ and $G(\bullet) := G(\bullet, T)$. Using the spectral projections $\mathcal{P}$ and $\mathcal{Q}$ we can decompose $v_n$ as $v_n = p_n + q_n$ where $p_n = \mathcal{P}v_n \in Y$ and $q_n = \mathcal{Q}v_n \in Z$ to obtain

$$\begin{aligned} p_{n+1} &= Lp_n + \mathcal{P}G(p_n + q_n), \\ q_{n+1} &= Lq_n + \mathcal{Q}G(p_n + q_n). \end{aligned} \tag{4.28}$$

This follows since $\mathcal{P}$ and $\mathcal{Q}$ commute with $L$ since they commute with $C$. This decomposition of the variable $v$ will also be useful to us when studying local unstable manifolds in Section 5. We now prove a lemma summarizing the important properties of $L$ and $G$ which we need.

**Lemma 4.11** *For any $a \in (0,1)$ there exists $T^* > 0$ such that for all $T \geq T^*$*

$$\begin{aligned} \|L^{-1}v\| &\leq a\|v\| \quad \forall v \in Y, \\ \|Lv\| &\leq a\|v\| \quad \forall v \in Z. \end{aligned} \tag{4.29}$$

*Furthermore, if Assumption* 4.5 *holds, then there exist* $K_i(t) > 0$, $i = 1, 2$ *such that*

$$\|\mathcal{R}[G(v,t) - G(w,t)]\| \leq K_1(t)k(\rho)\|v - w\|$$
$$\|\mathcal{R}G(v,t)\| \leq K_2(t)k(\rho)\rho \tag{4.30}$$

*for all* $v, w \in B(0,\rho)$ *and for* $\mathcal{R} = I, \mathcal{P}, \mathcal{Q}$, *where*

$$K_1(t), K_2(t) \to 0 \;\; as \;\; t \to 0 \tag{4.31}$$

*and*

$$\sup_{0 \leq t \leq 2T^*} K_1(t) = K_1 < \infty, \qquad \sup_{0 \leq t \leq 2T^*} K_2(t) = K_2 < \infty. \tag{4.32}$$

**Proof** To prove (4.29) apply Theorem 10.15 and choose $T^*$ such that $C_1 e^{-\gamma T^*} = a$ as in the derivation of (4.3). Now let $v, w \in B(0,\rho)$. Note that, by Lemma 2.4, we have that there exists $C = C(t) > 0$ such that

$$\|\bar{S}(t)v - \bar{S}(t)w\| \leq C(t)\|v - w\| \quad \forall v, w \in B(0,\rho); \tag{4.33}$$

here $C(0)$ is bounded and, without loss of generality, we may assume $C(t)$ is monotonically increasing in $t$.

To prove (4.30) note that, by Lemmas 10.6 and 10.13, Assumptions 4.5, (4.26) and the boundedness of the projections $\mathcal{P}, \mathcal{Q}$ there exists $K > 0$ such that

$$\|\mathcal{R}[G(v,t) - G(w,t)]\| \leq \int_0^t \frac{Kk(\rho)}{(t-s)^{\beta+\zeta}} \|[\bar{S}(s)v - \bar{S}(s)w]\| ds.$$

Thus we have from (4.33)

$$\|\mathcal{R}[G(v,t) - G(w,t)]\| \leq \frac{KC(t)t^{1-\beta-\zeta}k(\rho)}{(1-\beta-\zeta)} \|v - w\|. \tag{4.34}$$

A bound for $\|\mathcal{R}G(v)\|$ then follows in a similar fashion to the calculation of the Lipschitz constant in (4.34); we obtain

$$\|\mathcal{R}[G(v)]\| \leq \frac{Kt^{1-\beta-\zeta}k(\rho)\rho}{(1-\beta-\zeta)}. \tag{4.35}$$

The required result follows since $C(t)$ is bounded as $t \to 0_+$ and since $\beta + \zeta < 1$. □

**Remark** Note that $K_1(t)$ and $K_2(t)$ are $\mathcal{O}(t^{1-\beta-\zeta})$; if $\zeta + \beta > 0$ then this presents certain technical difficulties not present when considering ordinary differential equations — for many of the arguments we present it is not possible to let $t \to 0$. Instead it is neceesary to fix $t > 0$ and develop a separate argument to show that the objects of interest persist as $t \to 0$. □

For the remainder of Section 4, whenever estimating $v$ or its numerical counterpart, we employ the norm on $V$ given by

$$\|v\|_V = \max\{\|\mathcal{P}v\|, \|\mathcal{Q}v\|\}. \tag{4.36}$$

This choice of norm simplifies the exposition considerably since the majority of the estimation takes place either in $Y$ or $Z$ where $\|\bullet\|_V \equiv \|\bullet\|$. Note that

$$\|v\| \leq 2\|v\|_V. \tag{4.37}$$

Thus by (4.30), if $\|v\|_V \leq \varepsilon$ and $T \in [T^*, 2T^*]$, then

$$\begin{aligned} \|\mathcal{R}[G(v,t) - G(w,t)]\| &\leq K_1(t)k(2\varepsilon)\|v - w\| \\ \|\mathcal{R}G(v,t)\| &\leq 2K_2(t)k(2\varepsilon)\varepsilon \end{aligned} \tag{4.38}$$

In the remainder of this section, the ball $B(0,r)$, for any $r > 0$, is always measured in the $\|\bullet\|_V$ norm wherever the $v$ variable is under consideration.

We now seek a solution of (4.28) which satisfies the boundary conditions

$$p_m = \xi \in Y, \quad q_0 = \eta \in Z \tag{4.39}$$

where $\|\xi\|, \|\eta\| \leq \varepsilon/2$. Recalling that $v_n = p_n + q_n$, induction on (4.28) gives

$$\begin{aligned} p_n &= L^{n-m}p_m - \sum_{j=n}^{m-1} L^{n-1-j}\mathcal{P}G(v_j), \\ q_n &= L^n q_0 + \sum_{j=0}^{n-1} L^{n-1-j}\mathcal{Q}G(v_j). \end{aligned} \tag{4.40}$$

We wish to solve (4.40) subject to (4.39) for arbitrary $m > 0$. Such a solution corresponds to solving the eqn (4.2) with boundary conditions conditions specified in $Y$ at $t = mT$ and in $Z$ at $t = 0$; since $m$ is arbitrary, the time of flight between these points can be arbitrarily large. Finding all such solutions with $\varepsilon$ small corresponds to constructing the local phase portrait for equation (4.2) near $v = 0$. Figure 1 illustrates the situation pictorially. At time $t = 0$ each solution lies on the dotted line intersecting the $Z$-axis at $\eta$; at time $t = mT$ each solution lies on the dotted line intersecting the $Y$-axis at $\xi$. Four solutions are shown corresponding to four values of $m$: $m_1 \leq m_2 \leq m_3 \leq m_4$. As $m \to \infty$ the solution of (4.39, 4.40) gets closer and closer to the origin.

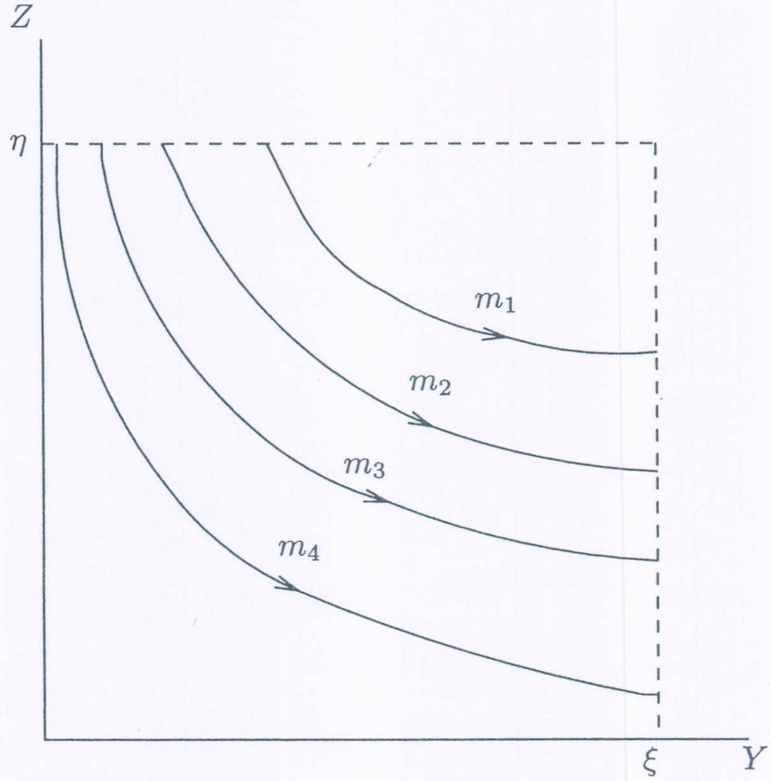


FIG. 1.

**Example 4.12** Consider the equation

$$v_t + Av = 2\pi^2 v$$

where $A$ is given by Example 2.1. In this case $C = A - 2\pi^2 I$ and $g(\bullet) \equiv 0$. Thus $C$ has eigenvalues $\{\lambda_m\}_{m=1}^{\infty}$, where $\lambda_m = m^2\pi^2 - 2\pi^2$, and corresponding eigenfunctions $\{\sin(m\pi x)\}_{m=0}^{\infty}$. Hence we have

$$\mathcal{P}V = \text{span}\{\sin(\pi x)\}, \quad \mathcal{Q}V = \text{span}\{\sin(2\pi x), \sin(3\pi x), \ldots\}.$$

The analog of (4.39) for this problem is

$$\mathcal{P}v(\tau) = \xi_1 \sin(\pi x), \quad \mathcal{Q}v(0) = \sum_{j=2}^{\infty} \eta_j \sin(j\pi x),$$

where $\tau = mT$. Separation of variables yields the unique solution

$$v(t) = \xi_1 e^{\pi^2(t-\tau)} \sin(\pi x) + \sum_{j=2}^{\infty} \eta_j e^{(2-j^2)\pi^2 t} \sin(j\pi x).$$

In the following we aim to generalize situations like the one shown in the previous example to incorporate the addition of the small nonlinear term $g(\bullet)$. Our approach to this problem will be to use a contraction mapping argument exploiting the fact that $e^{-Ct}$ is contractive on $Z$ and $e^{Ct}$ is contractive on $Y$, by virtue of Lemma 10.15 and (4.29). We need an appropriate functional setting for this argument and we let $\mathcal{V} = \{v_n\}_{n=0}^m$ denote an element of the product Hilbert space $\Psi = \{V\}^m$ and define

$$\|\mathcal{V}\|_\infty = \max_{0\le n\le m} \|v_n\|_V.$$

We also define the set

$$\Psi_\epsilon = \{\mathcal{V} \in \Psi : \|\mathcal{V}\|_\infty \le \varepsilon\}.$$

To study (4.40), (4.39) we use the contraction mapping theorem in $\Psi_\epsilon$. We generate iterates $\mathcal{V}^l = \{v_n^l\}_{n=0}^m$ through the definition $\mathcal{V}^{l+1} = M\mathcal{V}^l$ where

$$M\mathcal{V} = \{Mp_n + Mq_n\}_{n=0}^m$$

and where $Mp_n \in Y$ and $Mq_n \in Z$ are defined by

$$\begin{aligned} Mp_n &= L^{n-m}\xi - \textstyle\sum_{j=n}^{m-1} L^{n-1-j}\mathcal{P}G(v_j), \\ Mq_n &= L^n\eta + \textstyle\sum_{j=0}^{n-1} L^{n-1-j}\mathcal{Q}G(v_j). \end{aligned} \tag{4.41}$$

Clearly a fixed point of $M$ is a solution of (4.39), (4.40).

From the bounds (4.29) in Lemma 4.11 it follows that

$$\begin{aligned} \sum_{j=n}^{m-1} \|L^{n-j-1}v_j\| &\le \sum_{j=n}^{m-1} a^{1+j-n}\|v_j\| \le \sum_{l=1}^{m-n} a^l\|\mathcal{V}\|_\infty \le \frac{\|\mathcal{V}\|_\infty}{1-a} \qquad \forall v_j \in Y, \\ \sum_{j=0}^{n-1} \|L^{n-j-1}v_j\| &\le \sum_{j=0}^{n-1} a^{n-j-1}\|v_j\| \le \sum_{l=0}^{n-1} a^l\|\mathcal{V}\|_\infty \le \frac{\|\mathcal{V}\|_\infty}{1-a} \qquad \forall v_j \in Z. \end{aligned} \tag{4.42}$$

We may now prove

**Theorem 4.13** *Let Assumption 3.2 hold and let $a$ and $T^*$ be as in Lemma 4.11. Assume that $T \in [T^*, 2T^*]$ and that $\epsilon$ is chosen so that*

$$4\max\{K_1, 2K_2\}k(2\varepsilon) \le 1-a.$$

*Then, for any $m > 0$ and any $\xi \in Y$, $\eta \in Z$ with $\|\xi\|, \|\eta\| \le \frac{\varepsilon}{2}$ there exists a unique solution of (4.27) subject to (4.39) satisfying*

$$\max_{0\le n\le m} \|\mathbf{v}_n\|_V \le \varepsilon.$$

**Proof** Note that the bound on $\varepsilon$ in the theorem implies that

$$2\max\{K_1, 2K_2\}k(2\varepsilon) \le 1-a. \tag{4.43}$$

The bound has been chosen to be robust under doubling of $K_1$ and $K_2$ to allow for an analogous proof for the approximation of (2.9) under Assumptions 3.2.

Since (4.28) is equivalent to (4.40) we examine (4.40), (4.39). To prove the result we show that $M : \Psi_\epsilon \mapsto \Psi_\epsilon$ and is a contraction. From (4.41) we have, using (4.29), (4.38), (4.30) and (4.42), that

$$\begin{aligned}\|Mp_n\| &\leq \|L^{n-m}\xi\| + \sum_{j=n}^{m-1} \|L^{n-j-1}\mathcal{P}G(v_j)\| \\ &\leq \|\xi\| + \frac{1}{1-a}K_2 k(2\varepsilon)2\varepsilon.\end{aligned}$$

Hence, by (4.43) it follows that

$$\|Mp_n\| \leq \varepsilon \quad \forall n : 0 \leq n \leq m.$$

Likewise it may be shown that

$$\|Mq_n\| \leq \varepsilon \quad \forall n : 0 \leq n \leq m$$

so that $M\mathcal{V} \in \Psi_\epsilon$.

To show that the mapping contracts, consider (4.41) with $p_n \mapsto x_n$, $q_n \mapsto y_n, v_n \mapsto w_n$, define $w_n = x_n + y_n$ and set $\Omega = \{w_n\}_{n=0}^m$. Then, using (4.29), (4.30) and (4.42) we obtain from (4.41)

$$\|Mp_n - Mx_n\| \leq \frac{1}{1-a}K_1 k(2\varepsilon)\|\mathcal{V} - \Omega\|_\infty \quad \forall n : 0 \leq n \leq m$$

and

$$\|Mq_n - Mz_n\| \leq \frac{1}{1-a}K_1 k(2\varepsilon)\|\mathcal{V} - \Omega\|_\infty \quad \forall n : 0 \leq n \leq m.$$

Thus it follows from (4.43) that

$$\|\mathcal{V}^{k+1} - \Omega^{k+1}\|_\infty \leq \frac{1}{2}\|\mathcal{V}^k - \Omega^k\|_\infty.$$

Hence $M : \Psi_\varepsilon \mapsto \Psi_\varepsilon$ is a contraction and the result follows. □

We now use the previous lemma to solve eqn (4.2), and hence (2.9), subject to specified boundary conditions. Consider the problem

$$u_t + Au = f(u), \quad \mathcal{P}[u(\tau) - \bar{u}] = \xi, \; \mathcal{Q}[u(0) - \bar{u}] = \eta. \tag{4.44}$$

**Corollary 4.14 (Phase Portraits)** *Let Assumption 4.5 hold. Assume that $\epsilon$ is chosen so that*

$$4\max\{K_1, 2K_2\}k(2\varepsilon) \leq 1 - a.$$

*Then, for any $\tau \geq T^*$ and any $\xi \in Y$, $\eta \in Z$ with $\|\xi\|, \|\eta\| \leq \frac{\varepsilon}{2}$ there exists a solution $u(t)$, unique in $\overline{B(\bar{u}, 2\varepsilon)}$, of (4.44).*

**Proof** Note that existence and uniqueness for (4.44) is equivalent to the following problem with which we work during the course of the proof:

$$v_t + Cv = g(v), \quad \mathcal{P}v(\tau) = \xi, \ \mathcal{Q}v(0) = \eta. \tag{4.45}$$

The existence of a solution of (4.45) for $\tau = mT$ with $T \in [T^*, 2T^*]$ and any integer $m$ follows from Theorem 4.13. Since any real number larger than $T^*$ can be expressed as an integer multiple of a number in the interval $[T^*, 2T^*]$ the existence result follows. Assume for the purposes of contradiction that this solution of (4.45) is not unique in $B(0, \varepsilon)$. Then any other solution also generates a solution of (4.28) subject to (4.39); by the uniqueness of Theorem 4.13 the second solution must agree with the first at the points $t = nT$ and hence, by uniqueness for the initial value problem, they must agree everywhere. Uniqueness for (4.45) follows. By (4.37) the constant 2 appears when changing norms from $v$ to $u$. □

We now extend this analysis to the approximate semigroup $S^h(t)$ and prove an approximation result for the trajectories constructed in Corollary 4.14 which is independent of the time $\tau$. We start by converting the approximation to new coordinates and introduce the semigroup $\bar{S}^h(t) : V \mapsto V$ defined by

$$\bar{S}^h(t)v = S^h(t)(\bar{u} + v) - \bar{u}. \tag{4.46}$$

Compare this with (4.14). Here $v = \bar{u}^h - \bar{u}$ is a fixed point of $\bar{S}^h(t)$ for all $t > 0$; it is simply the equilibrium given by Theorem 4.3 in the new coordinates. We define the approximation error in the new coordinates by

$$\bar{E}(v; t) = \bar{S}(t)v - \bar{S}^h(t)v. \tag{4.47}$$

The Fréchet derivative of $\bar{E}(v; t)$ with respect to $v$ exists by Assumptions 3.2. When evaluated at a point $w \in V$, we denote it by $d\bar{E}(w; t)$. The following lemma is a straightforward consequence of Assumptions 3.2.

**Lemma 4.15** *For all $v \in B(0, R)$ and all $t \in \mathbf{S}$, $t > 0$, there exist constants $C_i = C_i(t, R) < \infty, i = 1, 2$, and a function $\kappa : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that*

$$\|\bar{E}(v; t)\| \leq C_1 h$$

$$\|d\bar{E}(v; t)\| \leq C_2 \kappa(h),$$

*where $\kappa(h) \to 0$ as $h \to 0_+$.*

Now we introduce the notation $V_n = \bar{S}^h(t_n)V_0$ where $t_n = nT$; this is analogous to the notation $v_n = \bar{S}(t_n)v_0$ used in the construction of phase portraits for (4.2). Note that

$$V_{n+1} \quad = \quad \bar{S}^h(T)V_n = \bar{S}(T)V_n - \bar{E}(V_n; T)$$

$$= \quad LV_n + G(V_n) - \bar{E}(V_n;T).$$

Hence we may write

$$V_{n+1} = LV_n + \tilde{G}(V_n) \tag{4.48}$$

where

$$\tilde{G}(v) = G(v) - \bar{E}(v;T). \tag{4.49}$$

Recall $T^*, K_1, K_2$ from Lemma 4.11. The following properties of $\tilde{G}$ will be needed:

**Lemma 4.16** *For any $\rho > 0$ there exists $h^* = h^*(\rho) > 0$ such that for all $h \leq h^*$ and $T \in [T^*, 2T^*]$*

$$\begin{aligned} \|\mathcal{R}[\tilde{G}(v) - \tilde{G}(w)]\| &\leq 2K_1 k(\rho)\|v - w\|, \\ \|\mathcal{R}\tilde{G}(v)\| &\leq 2K_2 k(\rho)\rho \end{aligned} \tag{4.50}$$

*for all $v, w \in B(0, \rho)$.*

**Proof** From (4.30), (4.32), (4.49), Lemma 4.15 and Theorem 11.4 we have

$$\begin{aligned} \|\mathcal{R}[\tilde{G}(v) - \tilde{G}(v)]\| &\leq \|\mathcal{R}[G(v) - G(w)]\| + \|\mathcal{R}[\bar{E}(v;T) - \bar{E}(w;T)]\| \\ &\leq K_1 k(\rho)\|v - w\| + \|\mathcal{R}\textstyle\int_0^1 d\bar{E}(sv + (1-s)w;T)[v - w]ds\| \\ &\leq K_1 k(\rho)\|v - w\| + C\kappa(h)\|v - w\|. \end{aligned}$$

By choice of $h$ sufficiently small the first result follows. The second result may be proved similarly. □

By applying appropriate projections to (4.48) we find that

$$\begin{aligned} P_{n+1} &= LP_n + \mathcal{P}\tilde{G}(P_n + Q_n), \\ Q_{n+1} &= LQ_n + \mathcal{Q}\tilde{G}(P_n + Q_n), \end{aligned} \tag{4.51}$$

where $P_n = \mathcal{P}V_n \in Y$ and $Q_n = \mathcal{P}V_n \in Z$. Using Lemma 4.16 we prove existence of solutions to (4.48) subject to

$$P_m = \xi \in Y, \quad Q_0 = \eta \in Z. \tag{4.52}$$

**Theorem 4.17** *Let Assumption 4.5 hold. Assume that $T \in [T^*, 2T^*]$ and that $\epsilon$ is chosen so that*

$$4\max\{K_1, 2K_2\}k(2\varepsilon) \leq 1 - a.$$

*Then if $h \in (0, h^*)$, where $h^* = h^*(\varepsilon)$ is given by Lemma 4.16, there exists $C > 0$ such that, for any $m > 0$ and any $\xi \in Y$, $\eta \in Z$ with $\|\xi\|, \|\eta\| \leq \frac{\varepsilon}{2}$*

*there exists a unique solution of (4.48) subject to (4.52), satisfying*

$$\max_{0\le n\le m} \|V_n\| \le \varepsilon$$

*and, furthermore,*

$$\max_{0\le n\le m} \|v_n - V_n\| \le Ch,$$

*where $v_n$ is given by Theorem* 4.13.

**Proof** The problem under consideration is equivalent to finding a solution of (4.51) which satisfies the boundary conditions (4.52), where $\|\xi\|, \|\eta\| \le \varepsilon/2$ and for any $T \in [T^*, 2T^*]$. As for (4.28), induction on (4.51) gives

$$\begin{aligned} P_n &= L^{n-m}P_m - \textstyle\sum_{j=n}^{m-1} L^{n-1-j}\mathcal{P}\tilde{G}(V_j), \\ Q_n &= L^n Q_0 + \textstyle\sum_{j=0}^{n-1} L^{n-1-j}\mathcal{Q}\tilde{G}(V_j). \end{aligned} \tag{4.53}$$

Thus our objective is to solve (4.53) subject to (4.52).

By analogy with (4.41) we generate iterates $\mathcal{V}^l = \{v_n^l\}_{n=0}^m$ through the definition $\mathcal{V}^{l+1} = M_h\mathcal{V}^l$ where

$$M_h\mathcal{V} = \{M_hP_n + M_hQ_n\}_{n=0}^k$$

and where $M_hP_n \in Y$ and $M_hQ_n \in Z$ are defined by

$$\begin{aligned} M_hP_n &= L^{n-m}\xi - \textstyle\sum_{j=n}^{k-1} L^{n-1-j}\mathcal{P}\tilde{G}(V_j), \\ M_hQ_n &= L^n\eta + \textstyle\sum_{j=0}^{n-1} L^{n-1-j}\mathcal{Q}\tilde{G}(V_j). \end{aligned} \tag{4.54}$$

Clearly a fixed point of $M_h$ is a solution of (4.52), (4.53). The same proof as used in Theorem 4.13 proves the existence of a solution to this problem; note that that proof was constructed to be robust under enlargement of $K_1$ and $K_2$ by a factor of 2 and that comparison of Lemmas 4.11, 4.16 shows that this is all that is necessary to extend the proof.

It remains to find the error bound and for this we use the uniform contraction principle. Let $\{v_n\}_{n=0}^m$ denote the fixed point of $M$ found in Theorem 4.13 and let $\{V_n\}_{n=0}^m$ denote the fixed point of $M_h$. Since the contraction constant is $\frac{1}{2}$ for both $M$ and $M_h$ we obtain from Theorem 11.2

$$\max_{0\le n\le m} \|v_n - V_n\|_V \le 2 \sup_{\|\mathcal{V}\|_\infty \le \varepsilon} \|M\mathcal{V} - M_h\mathcal{V}\|_\infty.$$

Here we denote the elements of $\Psi_\epsilon$ by $\mathcal{V} = \{w_n\}_{n=0}^\infty$ where $w_n \in V$ and we set $w_n = x_n + y_n$ where $x_n \in Y$ and $y_n \in Z$. Then (4.41) and (4.54) show that

$$\|Mx_n - M_hx_n\| \le \sum_{j=n}^{m-1} \|L^{n-1-j}\mathcal{P}\bar{E}(w_n;T)\|.$$

Thus, by Lemma 4.15 and (4.42), we have

$$\|Mx_n - M_h x_n\| \le \sum_{j=n}^{m-1} a^{n-1-j} C(T)h \le \frac{C(T)}{1-a}h.$$

Note that the result is independent of $m$. A similar result holds for $\|My_n - M_h y_n\|$. Combining these we deduce that, since $C(T)$ may be assumed monotonic increasing for $T \ge T^*$,

$$\max_{0\le n\le k} \|v_n - V_n\|_V \le \frac{C(2T^*)}{1-a}h.$$

□

We now consider the following analog of (4.44): find $u^h(t) \in V$ such that

$$u^h(t) = S^h(t)u^h(0) \quad \mathcal{P}[u^h(\tau) - \bar{u}] = \xi, \; \mathcal{Q}[u^h(0) - \bar{u}] = \eta. \tag{4.55}$$

**Theorem 4.18 (Convergence of Phase Portraits)** *Let Assumption* 4.5 *hold. Assume also that $\epsilon$ is chosen so that*

$$4\max\{K_1, 2K_2\}k(2\varepsilon) \le 1 - a$$

*and $h \in (0, h^*)$, $h^* = h^*(\varepsilon)$ given by Lemma* 4.16. *Then there is a constant $C > 0$ such that, for any $\tau > T^*$ and any $\xi \in Y$, $\eta \in Z$ with $\|\xi\|, \|\eta\| \le \frac{\varepsilon}{2}$ there exists a solution $u^h(t)$, unique in $\overline{B(\bar{u}, 2\varepsilon)}$, of (4.55) satisfying*

$$\sup_{T^*\le t\le\tau} \|u(t) - u^h(t)\| \le Ch,$$

*where $u(t)$ is the solution of (4.44) given in Corollary* 4.14.

**Proof** Theorem 4.17 gives the existence of a unique solution in $B(\bar{u}, 2\varepsilon)$ the constant 2 appearing when changing norms by (4.37). The required error estimate at integer multiples of $T$ also follows; the error between these points can be obtained by use of the standard finite time error bound in Assumptions 3.2. □

Instead of (4.39) it is also of interest to study (4.28) subject to

$$\|v_n\|_V \le \epsilon \; \forall n \ge 0, \quad q_0 = \eta \in Z, \quad \|\eta\| \le \frac{\varepsilon}{2}. \tag{4.56}$$

This is equivalent to finding a solution of the problem

$$u_t + Au = f(u), \quad \mathcal{Q}[u(0) - \bar{u}] = \eta, \quad \|\eta\| \le \frac{\varepsilon}{2}, \quad \|u(t) - \bar{u}\|_V \le \varepsilon \; \forall t \ge 0. \tag{4.57}$$

The analogous problem for the approximation is

$$u^h(t) = S^h(t)u^h(0), \quad \mathcal{Q}[u^h(0)-\bar{u}] = \eta, \quad \|\eta\| \leq \frac{\varepsilon}{2}, \quad \|u^h(t)-\bar{u}\|_V \leq \varepsilon \; \forall t \geq 0. \tag{4.58}$$

It is straightforward to show that solutions of (4.57) (resp. (4.58)) must approach the equilibrium point $\bar{u}$ (resp. $\bar{u}^h$) as $t \to \infty$. Hence by solving these problems we are constructing a set of points known as the *stable set.* This set actually has a manifold structure but we will not use that fact here. By following the method of proofs of Corollary 4.14 and Theorem 4.18 we obtain

**Theorem 4.19 (Convergence of Stable Sets)** *Let Assumption* 4.5 *hold and let $a$ and $T^*$ be as in Lemma 4.11. Assume also that $\epsilon$ is chosen so that*

$$4 \max\{K_1, 2K_2\} k(2\varepsilon) \leq 1 - a$$

*and $h \in (0, h^*)$, $h^* = h^*(\varepsilon)$ given by Lemma* 4.16. *Then there is a constant $C > 0$ such that, for any $\tau > T^*$ and any $\eta \in Z$ with $\|\eta\| \leq \frac{\varepsilon}{2}$ there exist solutions of (4.57), (4.58), unique in $B(\bar{u}, 2\varepsilon)$, satisfying*

$$\sup_{T^* \leq t \leq \tau} \|u(t) - u^h(t)\| \leq Ch.$$

## 4.5 Bibliography

The persistence of hyperbolic equilibrium points under appropriate perturbations is a consequence of the implicit function theorem in a Hilbert space — see Chow and Hale [16], for example. Theorem 4.3 simply adapts the proof of the implicit function theorem to our semigroup formulation of the problem, in particular to allow for the fact that the perturbations in Assumption 3.2 are not bounded as $t \to 0$.

Uniformly valid error estimates for $t \in (0, \infty)$ when the true solution approaches an exponentially stable equilibrium point are proved for finite element approximations of reaction-diffusion equations and Navier–Stokes equations in Larsson [73] and Heywood and Rannacher [55] respectively, and for finite difference methods applied to a reaction diffusion equation in Sanz-Serna and Stuart [85]. In the context of ordinary differential equations such results may be found in Stetter [87].

The existence of phase portraits for nonlinear problems can be deduced in finite dimensions from the Hartman–Grobman theorem [53]. Some infinite-dimensional versions of the Hartman–Grobman theorem are also available — see, for example, Bates and Lu [6]. The proof given here is motivated by the construction of stable and unstable sets in Henry [54] and its generalization to the phase portraits of reaction-diffusion equations in Larsson and Sanz-Serna [75]. See also Alouges and Debussche [1]. The idea of using the uniform contraction principle to understand the effect of perturbations is pervasive in the theory of dynamical systems; see Hale [49], Henry [54], Chow and Hale [16], for example. In the context of ordinary differential equations and numerical approximation Beyn [8] was the first to study phase portraits. That work has been taken further by Garay [41] [42] [43] [44] [45] [46] [47] [48].

Although not covered here the properties of periodic solutions under approximation has also been widely studied — see Beyn [7], Braun and Hershenov [13], Doan [25], Eirola [27] and Pugh and Shub [82] for ordinary differential equations, Alouges and Debussche [2] and Titi [95] for partial differential equations. The idea of local phase portrait for periodic solutions, and the effect of perturbation, is also analysed by Alouges and Debussche [2].

# 5 Unstable manifolds

## 5.1 Introduction

In this section we study the unstable set of an equilibrium point — that is the set of points from which solutions defined backwards in time converge to the equilibrium point. In Section 5.2 we introduce the definitions and background theory sufficient for the remainder of Section 5. In this regard Lemma 5.2 is crucial since it shows how the unstable set may be broken into two parts, the first a local part (the local unstable manifold) near the equilibrium point and the second being found by evolving the boundary of the first part forward in time. The local part of the unstable set has a manifold structure — it can be represented as a graph. Theorem 5.3 is at the core of the construction of this graph whilst Corollary 5.4 shows that the graph has the desired properties. Again by use of the uniform contraction principle from Appendix B, we incorporate the effect of perturbation into the graph representing the local unstable manifold. In Section 5.4 we return to the second part of the unstable set and use its characterization as the evolution of the boundary of the local unstable manifold to estimate the effect of perturbation. See Theorem 5.7.

## 5.2 Background theory

We start with a precise definition of the objects of interest to us in Section 5.

**Definition 5.1** *The unstable set of an equilibrium point $\bar{u}$ of (2.9) is the set*

$$W^u(\bar{u}) := \{u_0 \in V : \text{ a negative orbit } \{\varphi(t), t \leq 0\} \text{ exists through } u_0 \text{ and } \varphi(t) \to \bar{u} \text{ as } t \to -\infty\}.$$

*The local unstable manifold of $\bar{u}$ is the set*

$$W^{u,\varepsilon}(\bar{u}) := \{u_0 \in W^u(\bar{u}) : \|u(t) - \bar{u}\| \leq \varepsilon \; \forall t \leq 0\}.$$

*Analogous definitions hold for $S^h(t)$ and the notation $({W_h}^{u,\varepsilon}(\bar{u}^h))$ ${W_h}^u(\bar{u}^h)$ is used to denote the (local) unstable set (manifold) of an equilibrium $\bar{u}^h$ satisfying $S^h(t)\bar{u}^h = \bar{u}^h \;\; \forall t > 0$.*

Note that the unstable set is invariant whilst the local unstable manifold is negatively invariant. For brevity, however, we will refer to the local unstable manifold as invariant in the following proofs. The basic idea of Section 5 is first to prove convergence of the local unstable manifold and then to use this as a building block to prove convergence of the global unstable set by means of a compactness argument. The following lemma is essential in this regard.

**Lemma 5.2**
*The unstable set $W^u(\bar{u})$ of (2.9) is invariant and, furthermore,*

$$W^u(\bar{u}) = W^{u,\varepsilon}(\bar{u}) \cup \bigcup_{t>0} S(t)\Gamma \tag{5.1}$$

*where*

$$\Gamma = W^{\bar{u},\varepsilon}(\bar{u}) \bigcap \partial B(\bar{u},\varepsilon). \tag{5.2}$$

*Furthermore, if $W^u(\bar{u})$ is contained in a bounded set $B \subset V$, then the set $\bigcup_{t\geq 0} S(t)\Gamma$ is relatively compact.*

**Proof** It follows from the definition that, if $u \in W^u(\bar{u})$ then, for every $\tau > 0$ there exists $v^\tau \in V$ such that

$$\begin{gathered} S(\tau)v^\tau = u \\ v^\tau \to \bar{u} \text{ as } \tau \to \infty. \end{gathered} \tag{5.3}$$

The converse is also true: if (5.3) holds for every $\tau > 0$ then $u \in W^u(\bar{u})$.

Let $u \in W^u(\bar{u})$. Then

$$S(\tau)S(t)v^\tau = S(\tau + t)v^\tau = S(t)u$$

and, since (5.3) holds, we deduce that $S(t)u \in W^u(\bar{u})$ so that $S(t)W^u(\bar{u}) \subseteq W^u(\bar{u})$. Furthermore, since $S(t)v^t = u$ we have that, for every $t > 0$ and every $\tau > t$, $S(\tau - t)v^\tau = v^t$. Thus, from (5.3), we deduce that $v^t \in W^u(\bar{u})$. Thus $W^u(\bar{u}) \subseteq S(t)W^u(\bar{u})$ and the first part of the proof is complete.

We now establish (5.1) and (5.2). First we show that

$$W^u(\bar{u}) \subseteq W^{u,\varepsilon}(\bar{u}) \cup \bigcup_{t>0} S(t)\Gamma. \tag{5.4}$$

Let $u \in W^u(\bar{u}) \backslash W^{u,\varepsilon}(\bar{u})$. If $u \notin \overline{B(\bar{u},\varepsilon)}$ then, since (5.3) holds it follows that $v^0 = u$ and, by continuity, there exists $t > 0$ such that $S(t)v^t = u$ and $v^t \in \Gamma$. On the other hand, if $u \in \overline{B(\bar{u},\varepsilon)}$ then $\exists t > 0, v^t \in V : S(t)v^t = u$ with $v^t \in \Gamma$ since otherwise we have $u \in W^{u,\varepsilon}(\bar{u})$. Thus (5.4) holds.

Now we show that

$$W^u(\bar{u}) \supseteq W^{u,\varepsilon}(\bar{u}) \cup \bigcup_{t>0} S(t)\Gamma.$$

If

$$u \in \bigcup_{t>0} S(t)\Gamma$$

then there exists $w \in \Gamma$ such that $S(t)w = u$. Furthermore, since $w \in W^{u,\varepsilon}(\bar{u})$ it follows that $u \in W^u(\bar{u})$ by (5.3) and the result is proved.

Finally we assume that $W^u(\bar{u})$ is contained in a bounded set $B \subset V$. Thus $S(t)\Gamma \subset B$. By applying the compactness implied by Assumptions 2.3 on overlapping intervals $[-T,T]$, $[0,2T]$, $[T,3T]$, $[2T,4T], \ldots$ and noting that $S(mT)\Gamma \in B$ for all integers $m \geq -1$ we deduce that there exists $K = K(B,T) > 0, \eta > \beta$ such that

$$|y|_\eta \leq K \quad \forall y \in \bigcup_{t\geq 0} S(t)\Gamma;$$

this establishes the relative compactness. □

### 5.3 Local unstable manifolds

Throughout the remainder of Section 5, Assumption 4.5 is assumed to hold. We consider the mapping (4.28) where $L$ and $G$ are given by (4.2), (4.26). Let

$$\bar{Y} = \{p \in Y : \|p\| \leq \varepsilon\}$$

and seek $\Phi \in C(\bar{Y}, Z)$ :

$$q_n = \Phi(p_n) \Longleftrightarrow q_{n+1} = \Phi(p_{n+1}) \quad \forall n : \|p_n\|, \|p_{n+1}\| \leq \varepsilon. \tag{5.5}$$

As we shall see, the graph of the function $\Phi$ gives the local unstable manifold of $\bar{u}$. We shall look for $\Phi$ lying in the space

$$\begin{aligned}\Gamma(\varepsilon, \alpha) = \{\Phi \in C(\bar{Y}, Z) : \|\Phi\|_C = \sup\nolimits_{p\in\bar{Y}} \|\Phi(p)\| \leq \varepsilon, \\ \|\Phi(p_1) - \Phi(p_2)\| \leq \alpha\|p_1 - p_2\| \quad \forall p_1, p_2 \in \bar{Y}\}.\end{aligned}$$

The subscript $C$ in the norm on $C(\bar{Y}, Z)$ is simply to denote the space of continuous functions in which $\Phi$ lies. Recalling $T^*$ from Lemma 4.11, we can now prove

**Theorem 5.3** *Let Assumption* 4.5 *hold and assume that* $T \in [T^*, 2T^*]$. *Then there exists* $\varepsilon^* > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon^*]$ *and for all* $\alpha \in [1, 2]$ *there exists a unique* $\Phi = \Phi_{\varepsilon,\alpha} \in \Gamma(\varepsilon, \alpha)$ *such that (5.5) holds for (4.28) and, furthermore,*

$$\|q_m - \Phi(p_m)\| \leq \left(\frac{1+a}{2}\right)^m \|q_0 - \Phi(p_0)\| \tag{5.6}$$

*for all $m$ such that $v_n \in B(0,\varepsilon)$ for $1 \le n \le m$. Finally $\Phi_{\varepsilon,\alpha}$ is independent of $\varepsilon, \alpha$ in the sense that, if $\varepsilon_1 \le \varepsilon_2$ and $\alpha_1 \le \alpha_2$,*

$$\Phi_{\varepsilon_1,\alpha_1}(p) \equiv \Phi_{\varepsilon_2,\alpha_2}(p) \quad \forall p : \|p\| \le \varepsilon_1.$$

**Proof** We apply Theorem 12.3 with $r = \gamma = \varepsilon, b = a^{-1} > 1, \mu = (1 + a)/2, B_1 = 2K_1 k(2\varepsilon)$ and $B_2 = 4K_2 k(2\varepsilon)\varepsilon$. As in Section 4, $B_1$ and $B_2$ have been doubled to allow for the incorporation of perturbation error at a later stage.

Note that $r \ge (b-1)^{-1} B_2$ by choice of $\varepsilon$ sufficiently small, since $k(\varepsilon) \to 0$ as $\varepsilon \to 0$. Points (C1)–(C4) follow similarly. Since $\Gamma(\varepsilon_1, \alpha_1) \subseteq \Gamma(\varepsilon_2, \alpha_2)$ if $\varepsilon_1 \le \varepsilon_2$ and $\alpha_1 \le \alpha_2$, the independence of $\Phi$ from $\varepsilon$ and $\alpha$ follows from the uniqueness implied by the contraction mapping principle. □

**Corollary 5.4 (Local Unstable Manifold)** *Let Assumption (4.5) hold. Then there exists $\varepsilon_c > 0$ such that, for all $\varepsilon \in (0, \varepsilon_c]$, the origin for (4.2) has local unstable manifold representable as*

$$W^{u,\varepsilon}(0) = \mathcal{M} := \{v \in V : \mathcal{P}v \in \bar{Y}, \mathcal{Q}v = \Phi(\mathcal{P}v)\}$$

*where $\Phi \in \Gamma(\varepsilon, 1)$ satisfies (5.5) and (5.6).*

**Proof** For this proof it is necessary simply to verify: (i) that the manifold constructed by means of the graph in Theorem 5.3 is actually invariant for all $t > 0$ and not just $T \in [T^*, 2T^*]$; (ii) that the invariant manifold is the unstable manifold.

For (i) note that $\Phi$ is constructed as a fixed point of the mapping $T$ given by

$$\begin{aligned} p &= L\xi + \mathcal{P}G(\xi + \Psi(\xi)), \\ (T\Psi)(p) &= L\Psi(\xi) + \mathcal{Q}G(\xi + \Psi(\xi)). \end{aligned} \tag{5.7}$$

Note that $T$ depends on a parameter $t$ and when this dependence is important we shall denote it by $T^{(t)}$. Using the fact that $L$ and $G$ are constructed through one parameter semigroups $e^{-Ct}$ and $\bar{S}(t)$ it may be shown that $T^{(t)} \bullet T^{(s)} = T^{(s)} \bullet T^{(t)}$. Hence the fixed point constructed in Theorem 5.3 is independent of $t \in [T^*, 2T^*]$ by Theorem 11.3(i). To apply Theorem 11.3(ii) and deduce invariance for all sufficiently small $t$ it is sufficient to prove that

$$T^{(t)}\Phi_{\varepsilon_c,1} \in \Gamma(\varepsilon^*, 2), \tag{5.8}$$

provided that $\varepsilon_c < \varepsilon^*$. Straightforward analysis of (5.7) for $t$ sufficiently small shows that (5.8) holds by continuity.

For (ii), to show that the graph of $\Phi$ defines the local unstable manifold, consider the map

$$p_{n+1} = Lp_n + \mathcal{P}G(p_n + \Phi(p_n)).$$

It is straightforward to show that $\Phi(0) = 0$ by performing the contraction argument used in the construction of $\Phi$ in Theorem 12.3 in the space $\Gamma$ appended with the condition $\Psi(0) = 0$; it is necessary to use the fact that $G(0) = 0$. Thus, recalling $B_1 = 2K_1k(2\varepsilon)$ from Theorem 5.3 we have

$$\|p_n\| \leq a\|p_{n+1}\| + 2K_1(1+\alpha)k(2\varepsilon)\|p_n\|.$$

Hence, for $\varepsilon$ sufficiently small,

$$\|p_n\| \leq \frac{1+a}{2}\|p_{n+1}\|.$$

Thus, if $p_0 \in \bar{Y}$ then $p_n \in \bar{Y}$ for all $n \leq 0$ so that, by induction, $\|p_n\| \to 0$ as $n \to \infty$. Since $\Phi(0) = 0$ it follows that $v_n \to 0$ as $n \to -\infty$ for all $v_0$ on the invariant manifold. □

Now we consider the effect of approximation on the invariant manifold. Specifically we study (4.51) and try to find $\Phi \in C(\bar{Y}, Z)$ such that

$$Q_n = \Phi^h(P_n) \Longleftrightarrow Q_{n+1} = \Phi^h(P_{n+1}) \quad \forall n : \|P_n\|, \|P_{n+1}\| \leq \varepsilon. \tag{5.9}$$

**Theorem 5.5** *Let Assumption* 4.5 *hold and assume that* $T \in [T^*, 2T^*]$. *Then there exists* $\varepsilon^* > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon^*]$ *and for all* $\alpha \in [1,2]$ *there exists a unique* $\Phi^h = \Phi^h_{\varepsilon,\alpha} \in \Gamma(\varepsilon, \alpha)$ *such that (5.9) holds for (4.51) and, furthermore,*

$$\|Q_m - \Phi^h(P_m)\| \leq \left(\frac{1+a}{2}\right)^m \|Q_0 - \Phi(P_0)\| \tag{5.10}$$

*for all* $m$ *such that* $V_n \in B(0,\varepsilon)$ *for* $n = 0, \ldots, m$. *Futhermore* $\Phi^h_{\varepsilon,\alpha}$ *is independent of* $\varepsilon, \alpha$ *in the sense that, if* $\varepsilon_1 \leq \varepsilon_2, \alpha_1 \leq \alpha_2$,

$$\Phi_{\varepsilon_1,\alpha_1}(p) \equiv \Phi_{\varepsilon_2,\alpha_2}(p) \quad \forall p : \|p\| \leq \varepsilon_1.$$

*Finally* $\Phi^h$ *is close to* $\Phi$ *from Theorem* 5.5 *in the sense that there exists* $K > 0$ *such that*

$$\sup_{p \in \bar{Y}} \|\Phi(p) - \Phi^h(p)\|_C \leq Kh. \tag{5.11}$$

**Proof** Using Lemma 4.16 we deduce that the same method can be used to construct the local unstable manifold as that leading to Corollary 5.4 — note that the proof of Theorem 5.3 was constructed to be robust under enlargement of $K_1$ and $K_2$ by a factor of 2 and this is all that is required by comparing Lemmas 4.11, 4.16.

It remains to establish convergence of the graphs. The graph $\Phi$ is a fixed point of $T$ given by (5.7). Likewise $\Phi^h$ is a fixed point of $T^h$ given by

$$\begin{aligned} P &= L\xi + \mathcal{P}\tilde{G}(\xi + \Psi(\xi)), \\ (T^h\Psi)(P) &= L\Psi(\xi) + \mathcal{Q}\tilde{G}(\xi + \Psi(\xi)), \end{aligned} \tag{5.12}$$

where $\tilde{G}$ satisfies (4.49).

We now establish convergence. By the uniform contraction principle we have that, for $\Gamma = \Gamma(\varepsilon, \alpha)$,

$$\|\Phi - \Phi^h\|_C \leq \sup_{\Psi \in \Gamma} \tfrac{2}{1+a} \|T\Psi - T^h\Psi\|_C.$$

Thus it remains to estimate $T - T^h$. Clearly

$$\|(T^h\Psi)(P) - (T\Psi)(P)\| \leq \|(T^h\Psi)(P) - (T\Psi)(p)\| + \|(T\Psi)(p) - (T\Psi)(P)\|.$$

Since $T\Psi \in \Gamma$ we deduce that

$$\|(T^h\Psi)(P) - (T\Psi)(P)\| \leq \|(T^h\Psi)(P) - (T\Psi)(p)\| + \alpha\|p - P\|.$$

Comparing (5.7) and (5.12) and applying Lemma 4.16 we deduce that

$$\|(T^h\Psi)(P) - (T\Psi)(P)\| \leq Ch$$

and the result follows. □

**Corollary 5.6 (Convergence of Local Unstable Manifold)** *Let 4.5 hold. Then there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$ and all $\alpha \in [1, 2]$ the fixed point $\bar{u}^h - \bar{u}$ for $\bar{S}^h(t)$ has local unstable manifold representable as*

$$W^{u,\varepsilon}(\bar{u}^h - \bar{u}) = \mathcal{M}^h := \{v \in V : \mathcal{P}v \in \bar{Y}, \mathcal{Q}v = \Phi^h(\mathcal{P}v)\}$$

*where $\Phi^h \in \Gamma(\varepsilon, 1)$ and satisfies (5.9) and (5.10). Furthermore, there exists $C > 0$ such that, for any $\varepsilon$ sufficiently small, there exists a positive $\delta' < \delta$ such that*

$$dist\{W_{\mathrm{h}}{}^{u,\delta}(\bar{u}^h - \bar{u}), W^{u,\delta'}(0)\} \leq Ch,$$
$$dist\{W^{u,\delta}(0), W_{\mathrm{h}}{}^{u,\delta'}(\bar{u}^h - \bar{u})\} \leq Ch.$$

**Proof** Existence and convergence of a manifold invariant under the time $T$ map follows from Theorem 5.5. It is necessary to show that the graph constructed for $T \in [T^*, 2T^*]$ is invariant for all $t > 0$. To do this it is sufficient to show invariance for all $t$ sufficiently small. The same method as used in Corollary 5.4 cannot be used since Lemma 4.16 is not valid for $T \notin [T^*, 2T^*]$. We proceed by using a contradiction argument instead. Assume that $q(0) = \Phi(p(0))$ for some $p(0), q(0)$ such that $v(0) = p(0) + q(0)$ and consider the solution of $v(\tau) = S^h(\tau)v(0)$. Assume for contradiction that, for each $\tau \in (0, T)$, there exists $\eta = \eta(\tau) > 0$ such that

$$\|q(\tau) - \Phi^h(p(\tau))\| = \eta, \tag{5.13}$$

where $p(0), p(\tau) \in \bar{Y}$. Since $v(0) \in \mathcal{M}^h$ it follows that $v(-nT) \in \mathcal{M}^h$ for all positive integers $n$ provided $p(-nT) \in \bar{Y}$. Furthermore, if $p_n = p(nT)$ then, by (4.51),

$$\|p_n\| \leq a\|p_{n+1}\| + 2aB_2,$$

where $B_2 = 4K_2k(2\varepsilon)\varepsilon$ as in Theorem 5.3. Hence we deduce that

$$\|p_{-n}\| \le a^n\|p_0\| + (1-a^n)\frac{2aB_2}{1-a}.$$

Hence, for $\varepsilon$ so small that $2aB_2 \le (1-a)\|p_0\|$, we deduce that $p_{-n} \in \bar{Y}$ for all $n \ge 0$, so that $v(-nT) \in \mathcal{M}^h$ for all $n \ge 0$. Now let $t = -nT + \tau$ and note that, since $q(-nT) = \Phi^h(p(-nT))$, we have

$$\begin{aligned}
\|q(t) - \Phi^h(p(t))\| &\le \|q(-nT) - \Phi^h(p(-nT))\| + \|q(-nT) - q(t)\| \\
&\quad + \|\Phi^h(p(-nT)) - \Phi^h(p(t))\| \\
&\le \|q(-nT) - q(t)\| + \alpha\|p(-nT) - p(t)\| \\
&\le \max\{1, \alpha\}\|v(-nT) - v(t)\|_V \\
&\le C\|v(-nT) - v(t)\|.
\end{aligned}$$

But $\|v(t) - v(-nT)\| \le K(t, \|v(-nT)\|)$. Thus

$$\|q(t) - \Phi^h(p(t))\| \le K(t, \|v(-nT)\|).$$

Also

$$\begin{aligned}
\|v(-nT)\| &\le K\|v(-nT)\|_V \\
&\le K\max\{\|p(-nT)\|, \|q(-nT)\|\} \qquad (5.14) \\
&\le K\max\{\|p(-nT)\|, \varepsilon\}.
\end{aligned}$$

Since $\|p(-nT)\|$ is bounded uniformly in $n$ in terms of $\|p(0)\|$ and $\tau \in (0, T)$, we obtain

$$\|q(t) - \Phi^h(p(t))\| \le C(T, \|p_0\|).$$

Now let $t_m = -mT + \tau$; since $\tau$ may be arbitrarily small and since $v(-mT) \in B(0, \varepsilon)$ we can ensure $v(t_m) \in B(0, \varepsilon^*)$ provided $\varepsilon < \varepsilon^*$. Thus, by (5.6),

$$\|q(\tau) - \Phi^h(p(\tau))\| \le \left(\frac{1+a}{2}\right)^n \|q(t) - \Phi^h(p(t))\|$$

since $\tau = t + nT$ and since the manifold is attractive. Thus, by choice of $n$ sufficiently large we obtain a contradiction to (5.13). Hence $\Phi^h$ is invariant under $S^h(t)$ for all $t > 0$. The closeness follows from (5.11). □

## 5.4 Global unstable sets

In this section we examine the continuity of the global unstable sets of (2.9) with respect to numerical perturbation using our knowledge about the effect of perturbation on the local unstable manifold.

For the following theorem note that boundedness of the unstable manifold implies relative compactness by Lemma 5.2 so that the hypotheses could be weakened.

**Theorem 5.7 (Lower Semicontinuity of the Unstable Set)** *Assume that (4.2) has an equilibrium point $\bar{v}$ and that $\bar{v}^{\mathrm{h}}$ is the equilibrium point of $S^h(t)$ which converges to $\bar{v}$ as $h \to 0$. Further, suppose that Assumptions 4.5 holds. Then, if $W^u(\bar{v})$ is contained in a compact set $B \subset V$, it follows that*

$$dist(\overline{W^u(\bar{v})}, \overline{W_{\mathrm{h}}{}^u(\bar{v}^{\mathrm{h}})}) \to 0 \quad as \quad h \to 0.$$

**Proof** It is sufficient to prove that, given any $\varepsilon > 0$, there exists $\Delta > 0$ such that for every $y \in W^u(\bar{v})$ there exists $y^{\mathrm{h}} \in W_{\mathrm{h}}{}^u(\bar{v}^{\mathrm{h}})$ with the property that $\|y - y^{\mathrm{h}}\| \leq 2\varepsilon$ for $h \in (0, \Delta]$.

Recall $\partial B(\bar{v}, r)$ and $\Gamma$ given by (2.6), (5.1) and (5.2). Now set

$$\mathcal{W} = W^u(\bar{v}) \backslash W^{u,\varepsilon}(\bar{v}). \tag{5.15}$$

Then, for $\varepsilon$ sufficiently small,

$$\mathcal{W} = \bigcup_{t>0} S(t)\Gamma,$$

by Lemma 5.2. It follows that $\overline{\mathcal{W}}$ is compact by assumption. Note that $\{B(x;\epsilon): x \in \mathcal{W}\}$ is an $\varepsilon$-cover for $\overline{\mathcal{W}}$ and hence, since $\overline{\mathcal{W}}$ is compact, we may extract a finite subcover. Denote this subcover by $\{B_i(\varepsilon)\}_{i=1}^{I}$ and note that each $B_i(\varepsilon)$ contains a point $y_i \in \mathcal{W}$, where $B_i(\varepsilon) = B(y_i, \varepsilon)$. By construction there exists $x_i \in \Gamma$ and $T_i > 0$ such that $S(T_i)x_i = y_i$ for each $y_i \in \mathcal{W}$. Now, by Corallary 5.6, it follows that for any $\eta > 0$ there exists ${x^{\mathrm{h}}}_i \in W_{\mathrm{h}}{}^u(\bar{v}^{\mathrm{h}})$ and $\Delta(i) > 0$ such that

$$\|x_i - {x^{\mathrm{h}}}_i\| \leq \eta \quad \forall h \in (0, \Delta(i)]; \tag{5.16}$$

by the invariance of the unstable manifold (see Lemma 5.2) it follows that ${y^{\mathrm{h}}}_i = S^h(T_i){x^{\mathrm{h}}}_i \in W_{\mathrm{h}}{}^u(\bar{v}^{\mathrm{h}})$.

Note that

$$\begin{aligned} \|y_i - {y^{\mathrm{h}}}_i\| &= \|S(T_i)x_i - S^h(T_i){x^{\mathrm{h}}}_i\| \\ &\leq \|S(T_i){x^{\mathrm{h}}}_i - S^h(T_i){x^{\mathrm{h}}}_i\| + \|S(T_i){x^{\mathrm{h}}}_i - S(T)x_i\|. \end{aligned}$$

It now follows from Assumptions 3.2 and (5.16) that, by the continuity of

$S(T_i)$ and appropriate choice of $\eta$,

$$\|y_i - y^{\mathrm{h}}{}_i\| \le \frac{\varepsilon}{2} + C(T_i, \bar{v})\eta$$

for $h \in (0, \Delta(i)]$. Thus, by further reduction of $\Delta(i)$ if necessary, we find that

$$\|y_i - y^{\mathrm{h}}{}_i\| \le \varepsilon, \quad \forall h \in (0, \Delta(i)].$$

Since $I$ is finite, we deduce that there exists $\{y^{\mathrm{h}}{}_i\}_{i=1}^{I}$ each lying in $W_{\mathrm{h}}{}^u(\bar{v}^{\mathrm{h}})$ and $\Delta > 0$ such that

$$\max_{1 \le i \le I} \|y_i - y^{\mathrm{h}}{}_i\| \le \varepsilon \quad \forall h \in (0, \Delta].$$

Thus, since $y_i$ is the center of $B_i(\varepsilon)$, we deduce that for every $y \in B_i(\varepsilon)$ and $i$ such that $1 \le i \le I$ there exists $y^{\mathrm{h}}{}_i \in W_{\mathrm{h}}{}^u(\bar{v}^{\mathrm{h}})$ such that

$$\|y - y^{\mathrm{h}}{}_i\| \le 2\varepsilon \quad \forall h \in (0, \Delta].$$

Since the $B_i(\varepsilon)$, $i = 1, \ldots, I$, form a cover of $\overline{\mathcal{W}}$, we deduce that

$$\mathrm{dist}(\overline{\mathcal{W}}, W_{\mathrm{h}}{}^u(\bar{v}^{\mathrm{h}})) \le 2\varepsilon \quad \forall h \in (0, \Delta]. \tag{5.17}$$

Now, by Corollary 5.6 there exists $\delta' > 0$ such that

$$\mathrm{dist}(W^{u,\delta}(\bar{v}), W_{\mathrm{h}}{}^{u,\delta'}(\bar{v}^{\mathrm{h}})) \le 2\varepsilon \quad \forall h \in (0, \Delta], \tag{5.18}$$

possibly by further reduction of $\Delta$. Putting (5.17) and (5.18) together, the first result follows by (5.15). □

## 5.5 Bibliography

Two important approaches to the construction of unstable manifolds are the Lyapunov–Perron technique (basically the method used for stable sets in Section 4) and the Hadamard graph transform technique employed in this Section. See Babin and Vishik [4], Hale [50], Wells [97] and Wiggins [98] for material on unstable manifolds.

The existence and convergence of unstable manifolds for approximation of ordinary differential equations was first considered by Beyn [8]; the generalization to partial differential equations was studied by Alouges and Debussche [1] and by Larsson and Sanz-Serna [75], the latter using Lyapunov–Perron techniques. The approach given here is based on the Hadamard graph transform and is similar to the technique used to construct perturbations of center unstable manifolds of ordinary differential equations in Beyn and Lorenz [10].

# 6 Inertial manifolds

## 6.1 Introduction

The inertial manifold is a finite-dimensional set (in terms of the number of eigenfunctions of $A$ needed to represent it) which has a manifold structure and exponentially attracts all solutions of (2.9). Roughly speaking it exists provided the spectrum of $A$ has sufficiently large gaps compared to the size of the nonlinearity $F$. In Section 6.2 we make some precise definitions of the inertial manifold and, in Example 6.6, give an explicit construction of the inertial manifold for a non-local reaction-diffusion equation. Theorem 6.4 gives a construction of the inertial manifold for a mapping derived by considering the time $T$ flow of the semigroup and applying a contraction argument very similar to that used in Section 5 to construct the local unstable manifold; Theorem 6.5 shows that this inertial manifold is also invariant for the underlying partial differential equation. In Section 6.3 we incorporate the effect of the approximation, once again using the uniform contraction principle.

## 6.2 Existence theory

We assume that there exists $R > 0$ such that

$$\begin{aligned} &\exists T = T(\rho, R) : S(t)B(0,\rho) \subseteq B(0,R) \quad \forall t \geq T, \\ &S(t)B(0,R) \subseteq B(0,R) \quad \forall t \geq 0. \end{aligned} \tag{6.1}$$

Note that typical equations under consideration do not satisfy (2.11) uniformly in $V$ but only on bounded sets of $V$; use of conditions such as (6.1) can be used to achieve (2.11) by smooth modification of $F(\bullet)$ outside $B(0, R)$. Thus throughout Section 6 we will make the following assumption.

**Assumption 6.1** *The function $F$ in (2.9) satisfies (6.1) and (2.11).*

Recall that $A$ has eigenvalues $\lambda_j$ and eigenfunctions $\varphi_j$ ordered so that (2.1) holds. In this section we let $\mathbb{P}^m$ denote the projection of $X$ onto the first $m$ eigenfunctions of $A$ so that

$$v = \sum_{j=1}^{\infty} v_j \varphi_j \Rightarrow \mathbb{P}^m v = \sum_{j=1}^{m} v_j \varphi_j.$$

We also set $\mathbb{Q}^m = I - \mathbb{P}^m$ and $Y = \mathbb{P}^m X, Z = \mathbb{Q}^m X$.

**Definition 6.2** *An inertial manifold for a semigroup $S(t)$ is a positively invariant set $\mathcal{M}$, defined through a Lipschitz graph $\Phi : Y \mapsto Z$ and constant $R > 0$, and satisfying both of the following:*

*(i) $\mathcal{M}$ may be expressed in the form*

$$\mathcal{M} := \{u \in V : \mathbb{Q}^m v = \Phi(\mathbb{P}^m v)\} \bigcap B(0,R), \tag{6.2}$$

*(ii) there exists $\nu > 0$ such that, for each $u_0$ in $V$,*

$$\exists C = C(u_0) : dist(S(t)u_0, \mathcal{M}) \leq Ce^{-\nu t} \quad \forall t \geq 0. \tag{6.3}$$

The reason why inertial manifolds are of theortical importance is that they show that the large time behavior of certain partial differential equations is governed by a finite-dimensional ordinary differential equation. Specifically all solutions are attracted to $\mathcal{M}$ exponentially and, on $\mathcal{M}$, the solutions are governed by the equation

$$p_t + Ap = \mathbb{P}^m F(p + \Phi(p)).$$

This is equivalent to a system of $m$ ordinary differential equations in the $m$ coefficients of $p(t)$ in an expansion in terms of the $\varphi_i$'s.

**Example 6.3** Consider eqn (2.9) with $A$ given in Example 2.1 and $F$ given by $F(u)(x) := f(|u|^2)u(x)$ for some smooth $f$ satisfying

$$f(x) \leq \lambda \quad \forall x \in \mathbb{R}^+. \tag{6.4}$$

Recall that $|\bullet|$ denotes the norm on $L_2((0,1))$ for Example 2.1. Thus (6.4) is equivalent to the non-local reaction-diffusion equation

$$\begin{aligned} u_t = u_{xx} + f(\textstyle\int_0^1 u^2(s,t)ds)u, &\quad (x,t) \in (0,1) \times (0,\infty), \\ u(0,t) = u(1,t) = 0, &\quad t > 0, \\ u(x,0) = u_0(x). & \end{aligned} \tag{6.5}$$

Under further conditions on the behavior of $f$ at infinity, (2.11) will be satisfied. It will also be useful to consider the problem

$$\begin{aligned} v_t = v_{xx}, &\quad (x,t) \in (0,1) \times (0,\infty), \\ v(0,t) = v(1,t) = 0, &\quad t > 0, \\ v(x,0) = u_0(x). & \end{aligned} \tag{6.6}$$

Recall $\varphi_k$ defined in Example 2.1. We let $a(t) = |u(\cdot,t)|^2, b(t) = |v(\cdot,t)|^2$. If

$$u_0(x) = \sum_{j=1}^{\infty} a_k \varphi_k, \quad a_k = \langle u_0, \varphi_k \rangle$$

then separation of variables shows that

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{-k^2\pi^2 t} e^{\int_0^t f(a(s))ds} \varphi_k. \tag{6.7}$$

Here $a(t)$ satisfies the nonlinear integral equation

$$a(t) = \exp\left\{2\int_0^t f(a(s))ds\right\} b(t).$$

Now we let

$$\mathcal{M} = \{v \in V : \langle v, \varphi_k \rangle = 0 \quad \forall k \geq m\}$$

and show that $\mathcal{M}$ is an inertial manifold for appropriately chosen $m$. First note that $\mathcal{M}$ is invariant for any $m$ such that if $u_0(x) \in \mathcal{M}$ then $a_k = 0$ for all $k \geq m$; by (6.7) it follows that $u(x,t) \in \mathcal{M}$. It remains to show that $\mathcal{M}$ attracts exponentially. Note that from (6.4)

$$\int_0^t f(a(s))ds \leq \lambda t.$$

Hence

$$\langle u(x,t), \varphi_k \rangle = a_k e^{-k^2\pi^2 t} e^{\int_0^t f(a(s))ds} \leq a_k e^{(\lambda - k^2\pi^2)t}.$$

Choosing $m$ such that $\lambda - m^2\pi^2 \leq -\mu < 0$ we deduce that

$$\langle u(x,t), \varphi_k \rangle \leq a_k e^{-\mu t} \quad \forall k \geq m.$$

Since the $a_k$ are uniformly bounded in $k$ for initial data in $V \equiv H_0^1((0,1))$ it follows that $\mathcal{M}$ is exponentially attracting with rate at least $e^{-\mu t}$.

A similar explicit construction of an inertial manifold is given in Bloch and Titi [12]. That paper concerns a nonlinear beam equation where the nonlinearity occurs only through the appearance of the $L_2$-norm of the unknown. A closely related analysis to that given here allows for construction of an inertial manifold with structure similar to that given here.

Our basic approach is to use (2.15) to formulate a mapping and prove that the mapping has an inertial manifold; we then show that the inertial manifold for the map is also positively invariant under (2.9) and hence an inertial manifold for (2.9). We define

$$\begin{gathered} L(t) = e^{-At}, \quad G(u,t) = \int_0^t L(t-s)F(S(s)u)ds. \\ L := L(T), G(\bullet) := G(\bullet, T). \end{gathered} \tag{6.8}$$

Now consider the mapping

$$u_{n+1} = Lu_n + G(u_n), \tag{6.9}$$

where $u_n = S(nT)u_0$. We set $p_n = \mathbb{P}^m u_n, q_n = \mathbb{Q}^m u_n$ and seek an invariant manifold for (6.9) which hence satisfies

$$q_n = \Phi(p_n) \Longleftrightarrow q_{n+1} = \Phi(p_{n+1}) \quad \forall n \geq 0 \tag{6.10}$$

and is attractive in the sense that there exists $\mu \in (0,1)$ such that

$$\|q_n - \Phi(p_n)\| \leq \mu^n \|q_0 - \Phi(p_0)\| \quad \forall n \geq 0. \tag{6.11}$$

Let $\Gamma = \Gamma(\epsilon, \delta)$ denote the closed subset of $C(Y, Z)$ satisfying

$$\|\Psi\|_\Gamma := \sup_{p\in Y} \|\Psi(p)\| \leq \epsilon,$$

$$\|\Psi(p_1) - \Psi(p_2)\| \leq \delta\|p_1 - p_2\| \qquad \forall p_1, p_2 \in Y.$$

We may now prove the preliminary existence result for (6.9).

**Theorem 6.4** *Let Assumption* 6.1 *hold. Suppose that for any* $K_3, K_4 > 0$ *there exists an integer* $q_0 > 0$ *such that*

$$\lambda_{q+1}^{1-\beta} \geq K_3, \qquad \lambda_{q+1} - \lambda_q \geq K_4\lambda_{q+1}^\beta$$

*for all* $q \geq q_0$. *Then for any* $\zeta > 1$ *and* $\epsilon', \delta' > 0$, *there exists* $T > 0$, *integer* $q_0 > 0$ *and* $\mu \in (0,1)$ *such that, if* $q \geq q_0$, $\epsilon \in [\varepsilon', \zeta\varepsilon']$, $\delta \in [\delta', \zeta\delta']$ *then there exists* $\Phi \in \Gamma(\varepsilon, \delta)$ *such that (6.10), (6.11) holds for (6.9); thus there exists* $C = C(u_0) > 0$ *such that, if* $\mathcal{M}$ *is given by (6.2) with* $R$ *given by (6.1), then*

$$dist(u_n, \mathcal{M}) \leq C\mu^n. \tag{6.12}$$

**Proof** We show that the mapping (6.9) satisfies $(G1)$–$(G3)$ and $(C1)$–$(C4)$ of Theorem 12.3; the existence (6.10) and attractivity (6.11) then follow from Theorem 12.3 with $r = \infty$, $\gamma = \epsilon \in [\varepsilon', K\varepsilon']$, $\alpha = \delta \in [\delta', K\delta']$.

We define $\lambda = \lambda_q, \Lambda = \lambda_{q+1}$. Then $L$ satisfies $(G1), (G2)$ with

$$b = e^{-\lambda T}, \qquad a = e^{-\Lambda T}, \qquad c = e^{-\lambda_1 T}.$$

We set $\Lambda T = \sigma \in (0, \infty)$. From the definition of $G(u)$ we have, using Lemma 10.6 and Assumption 6.1 that there exists $C_1 > 0$ such that

$$\begin{aligned} \|G(u)\| &\leq \int_0^T |A^\beta L(T-s)F(S(s)u)|ds \\ &\leq \int_0^T \frac{CK}{(T-s)^\beta}ds \\ &\leq \frac{T^{1-\beta}}{1-\beta}C_1K. \end{aligned} \tag{6.13}$$

Similarly, using (2.16) (Lipschitz continuity of the semigroup $S(t)\bullet$, which follows from Assumption 6.1) and the construction of $F$, there exists $C_2 = C_2(T) > 0$ such that

$$\|G(u) - G(v)\| \leq \frac{T^{1-\beta}}{1-\beta}C_2K\|u - v\|.$$

Thus, $G(u)$ satisfies $(G3)$ with

$$B_1 = B_2 = 2T^{1-\beta}K_5, \tag{6.14}$$

where $K_5 = K\max\{C_1, C_2\}/(1-\beta)$. We have doubled $B_1, B_2$ to allow for incorporation of perturbation error in Section 6.3.

We verify $(C1)$–$(C4)$. Let $\epsilon, \delta > 0$, $\sigma \in (0,\infty)$, $\mu \in (e^{-\sigma}, 1)$ be given. Define $K_3, K_4$ and $T$ by

$$T = \frac{\sigma}{\Lambda}$$

$$K_3 = 2\max\left\{\frac{e^{\sigma}K_5\sigma^{1-\beta}(1+\delta)}{\mu - e^{-\sigma}}, \frac{(1+\sigma)K_5}{\epsilon\sigma^{\beta}}\right\}, \tag{6.15}$$

$$K_4 = \frac{K_5(1+\delta)^2 e^{\sigma}}{\delta\sigma^{\beta}}. \tag{6.16}$$

Choose $q$ such that

$$\Lambda^{1-\beta} \geq K_3, \quad \Lambda - \lambda \geq K_4\Lambda^{\beta}. \tag{6.17}$$

By (6.17) and (6.15) we deduce that $\Lambda^{1-\beta} \geq 2e^{\sigma}K_5\sigma^{1+\beta}(1+\delta)/\mu$. Putting $\sigma = \Lambda T$ and $\Lambda^{1-\beta} = (\sigma/T)^{1-\beta}$ we obtain

$$2e^{\Lambda T}K_5T^{1-\beta}(1+\delta) \leq \mu.$$

Since $\lambda \leq \Lambda$, we have

$$2e^{\lambda T}K_5T^{1-\beta}(1+\delta) \leq \mu;$$

this is simply $b^{-1}B_1(1+\delta) \leq \mu$, as required to establish $(C1)$.

By (6.16) and (6.15) we deduce that $\Lambda^{1-\beta} \geq 2(1+\sigma)K_5/(\sigma^{\beta}\epsilon)$ so that, putting $1+\sigma = 1+\Lambda T$, we have

$$2(1+\Lambda T)K_5T\frac{\Lambda^{\beta}}{\sigma^{\beta}} \leq \epsilon\Lambda T.$$

Adding $\epsilon$ to both sides and dividing by $1+\Lambda T$, we obtain

$$\frac{\epsilon}{1+\Lambda T} + 2K_5T^{1-\beta} \leq \epsilon.$$

Since $e^{-x} \leq 1/(1+x)$ for all $x > 0$, we have $a\epsilon + B_2 \leq \epsilon$ and $(C2)$ is established.

To prove $(C3)$ we note that, by (6.17),

$$\Lambda - \lambda \geq K_4\Lambda^{\beta} \geq 2K_5(1+\delta)^2e^{\sigma}\frac{\Lambda^{\beta}}{\delta\sigma^{\beta}}.$$

Since $e^x - 1 \geq x$ for positive $x$, we may bound $(\Lambda-\lambda)T$ by $e^{(\Lambda-\lambda)T} - 1$. Multiplying the previous inequality by $T$ and using $\sigma = \Lambda T$, we find

$$\delta e^{-\Lambda T} + 2K_5(1+\delta)^2T^{1-\beta} \leq \delta e^{-\lambda T}.$$

Since $(1+\delta)^2 = (1+\delta)+\delta(1+\delta)$, this last inequality becomes $\delta a + (1+\delta)B_1 \leq \delta b - \delta(1+\delta)B_1$, which is $(C3)$.

$(C4)$ is obtained by noting that, from (6.15),

$$\Lambda^{1-\beta} \geq K_3 \geq 2K_5\sigma^{1-\beta}\frac{(1+\delta)}{\mu - e^{-\sigma}}.$$

Rearranging this gives $2K_5T^{1-\beta}(1+\delta) + e^{-\Lambda T} \leq \mu$. Using the definitions of $a, B_1$, this last expression becomes $a + 2(1+\delta)B_1 \leq \mu$, which is $(C4)$.

Note that $q_0$ depends on $K_3$ and $K_4$, and hence on $\epsilon$ and $\delta$. Taking the supremum of $q$ over all $\epsilon \in [\epsilon', \zeta\epsilon']$, $\delta \in [\delta', \zeta\delta']$ yields $q_0$ such that the result holds.

It remains to establish (6.12). Recall that $u_n = p_n + q_n$. Note that

$$\mathrm{dist}(u_n, \mathcal{M}) \leq \|u_n - \tilde{u}_n\|$$

where $\tilde{u}_n = p_n + \Phi(p_n)$. Thus, by (12.4) of Theorem 12.3,

$$\mathrm{dist}(u_n, \mathcal{M}) \leq \|q_n - \Phi(p_n)\| \leq \mu^n \|q_0 - \Phi(p_0)\|$$

and the result follows since (6.1) of Assumption 6.1 ensures that $u_n$ enters $B(0, R)$ after a finite number of iterations. □

Theorem 6.4 shows that the time $T$ flow of the semigroup for (2.9) has an attractive invariant manifold. We show now that this manifold is in fact an inertial manifold for eqn (2.9).

**Theorem 6.5 (Inertial Manifolds)** *Under the assumptions of Theorem 6.4, eqn (2.9) has an inertial manifold $\mathcal{M}$ satisfying (6.2) and (6.3) for $\Phi \in \Gamma(\varepsilon', \delta')$ and $\mu > 0$ given in Theorem 6.4 and $R$ given by (6.1).*

**Proof** It remains to show that the manifold $\mathcal{M}$ constructed as a consequence of Theorem 6.4 is in fact invariant for the underlying partial differential equation, rather than just the time $T$ flow of the semigroup. To do this set $\Omega = S(\tau)\mathcal{M}$ for some $\tau \in (0, T)$. We show that, for all $\tau$ sufficiently small, $\Omega \equiv \mathcal{M}$ and hence that $\mathcal{M}$ is invariant for $S(t)$. The required result then follows.

Notice that $S(T)\Omega = S(T)S(\tau)\mathcal{M} = S(\tau)S(T)\mathcal{M} = S(\tau)\mathcal{M} = \Omega$ so that $\Omega$ is invariant under the time $T$ discrete map. In addition, we can show that $\Omega$ is the graph of a global function: recall the definitions of $L(t)$ and $G(u, t)$ given in (6.8); by applying the method of proof of Lemma 12.4 it follows that, for every $p \in Y$ there exists a unique $\xi$ so that

$$p = L(\tau)\xi + \mathbb{P}^m G(\xi + \Phi(\xi), \tau),$$

for any $\tau$ sufficiently small. Thus $\Omega$ can be expressed as a graph $q = \Psi(p)$ where $\Psi : Y \mapsto Z$ is given by

$$\Psi(p) := L(\tau)\Phi(\xi) + \mathbb{Q}^m G(\xi + \Phi(\xi), \tau).$$

Recall from Theorem 6.4 that $\Phi \in \Gamma(\delta', \epsilon')$. Similarly to the calculation of (6.13) we have

$$\begin{aligned}\|\Psi\| &\leq e^{-\Lambda\tau}\epsilon' + \frac{\tau^{1-\beta}}{1-\beta}C_1K \\ &\leq e^{\Lambda(T-\tau)}a\epsilon' + B\left(\frac{\tau}{T}\right)^{1-\beta}.\end{aligned}$$

Notice that for $\tau = 0, \tau = T$, $\|\Psi\| \leq \epsilon'$. In general, for $0 \leq \tau \leq T$ there is some $\eta \geq 0$ such that

$$\|\Psi\| \leq \epsilon' + \eta.$$

In a similar manner, one obtains

$$\|\Psi(p_1) - \Psi(p_2)\| \leq (\delta' + \eta)\|p_1 - p_2\|$$

for $0 \leq \tau \leq T$. Note that, by continuity, $\eta$ can be made arbitrarily small by requiring $\tau$ to be small. Thus we see that for any $\eta > 0$ there exists a $\tau^*(\eta) > 0$ such that

$$\Psi \in \Gamma(\epsilon' + \eta, \delta' + \eta)$$

for all $\tau \in (0, \tau^*(\eta))$. Thus we choose $\zeta$ so large that $(C1)$–$(C4)$, verified in the Theorem 6.4, hold for all $\delta \in [\delta', \delta' + \eta], \epsilon \in [\epsilon', \epsilon' + \eta]$. For such $\delta, \epsilon$ we have $\Psi \in \Gamma(\delta, \epsilon)$ and we may again apply Lemma 12.4 to obtain that for every $p \in Y$ there exists a $p_0 \in Y$ such that

$$\begin{aligned}p &= Lp_0 + \mathbb{P}^m G(p_0 + \Psi(p_0)), \\ q &= L\Psi(p_0) + \mathbb{Q}^m G(p_0 + \Psi(p_0)).\end{aligned}$$

However, since $S(T)\Omega = \Omega$, $u = p + q \in \Omega$ and $\Omega = \text{Graph}(\Psi)$, we must have $q = \Psi(p)$. Thus $\Psi$ is a fixed point of the map $T$ constructed in (12.8), and by the uniqueness of the fixed point under the conditions of Theorem 6.4, it follows that $\Psi \equiv \Phi$. One may now repeat the argument on the intervals $t \in (k\tau, (k+1)\tau)$ for integer $k \geq 1$. It follows that $\mathcal{M} = \text{Graph}(\Phi)$ is an invariant manifold for (2.9) for all time.

To see that this manifold exponentially attracts all solutions as in (6.3) let $t > 0$ and $u_0$ be given. Set $t_n(s) = nT + s, s \in [0, T]$. Then from (6.12)

$$\text{dist}(u(t_n), \mathcal{M}) = \text{dist}(S(nT)u(s), \mathcal{M}) \leq \mu^n C(|u(s)|_\gamma).$$

Since $u(s)$ depends continuously on $u(0)$ for all $s \in (0, T)$, the result follows by choosing $\nu$ appropriately. □

### 6.3 Perturbation theory

In the previous section an inertial manifold was constructed for (2.9) under (2.11) on $F$. Recall that in practice assumptions such as (2.11) only hold within a bounded set $B(0, R)$ satisfying (6.1); the function $F$ can then be modified outside $B(0, R)$ to ensure that (2.11) holds. We make similar assumptions about $S^h(t)$ : we assume that there exists $R > 0$ such that

$$\exists T = T(\rho, R) : S^h(t)B(0,\rho) \subseteq B(0,R) \quad \forall t \geq T,$$
$$S^h(t)B(0,R) \subseteq B(0,R) \quad \forall t \geq 0. \tag{6.18}$$

Thus throughout this section we make

**Assumption 6.6** *There exists $R > 0$ such that (6.18) holds for $S^h(t)$ of Assumption 3.2.*

We now define a new map

$$\tilde{S}^h(t)u := \theta(\|u\|^2)S^h(t)u + (1 - \theta(\|u\|^2))S(t)u, \tag{6.19}$$

where $\theta \in C^\infty(\mathbb{R}^+, \mathbb{R}^+)$ satisfies

$$\theta(x) = 1 \; \forall x : x^2 \leq R, \quad \theta(x) = 0 \; \forall x : x^2 \geq 2R$$

and $\theta'(x)$ is uniformly bounded. The derivative of $\tilde{S}^h(t)$ with respect to $u \in V$ and evaluated at $v \in V$ is denoted by $d\tilde{S}^h(v,t)$. Note that the new map does not satisfy the semigroup property on $V$ but, by virtue of (6.18), it does on $B(0,R)$ since it is coincident with $S^h(t)$ on that set. The reason for considering $\tilde{S}^h(t)$ is that it has been constructed to be $C^1$ close to $S(t)$ *uniformly* on $V$ :

**Lemma 6.7** *Define*

$$\tilde{E}(u;t) := S(t)u - \tilde{S}^h(t)u,$$
$$d\tilde{E}(u;t) := dS(u;t) - d\tilde{S}^h(u;t).$$

*Then for all $t \in \mathbf{S}$, $t > 0$, there exist constants $C_i(t) < \infty, i = 1, 2$, and a function $\kappa : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that the maps $S(\bullet)\bullet \in C^1(\mathbb{R}^+ \times V, V)$ and $\tilde{S}^h(\bullet)\bullet \in C^1(\mathbb{R}^+ \times V, V)$ satisfy*

$$\|\tilde{E}(u;t)\| \leq C_1 h \quad \forall u \in V,$$

$$\|d\tilde{E}(u;t)\| \leq C_2 \kappa(h) \quad \forall u \in V,$$

*where $\kappa(h) \to 0$ as $h \to 0_+$.*

We now let $U_n = \tilde{S}^h(nT)U_0$ so that

$$U_{n+1} = LU_n + \tilde{G}(U_n), \tag{6.20}$$

where

$$\tilde{G}(u) = G(u) + \tilde{E}(u;T). \tag{6.21}$$

We set $P_n = \mathbb{P}^m U_n, Q_n = \mathbb{Q}^m U_n$ and seek an invariant manifold for (6.20) which hence satisfies

$$Q_n = \Phi^h(P_n) \Longleftrightarrow Q_{n+1} = \Phi^h(P_{n+1}) \quad \forall n \geq 0 \tag{6.22}$$

and is attractive in the sense that

$$\|Q_n - \Phi(P_n)\| \leq \mu^n \|Q_0 - \Phi(P_0)\| \quad \forall n \geq 0. \tag{6.23}$$

**Theorem 6.8** *Let Assumptions* 6.1 *and* 6.6 *hold. Suppose that for any* $K_3, K_4 > 0$ *there exists an integer* $q_0 > 0$ *such that*

$$\lambda_{q+1}^{1-\beta} \geq K_3, \qquad \lambda_{q+1} - \lambda_q \geq K_4 \lambda_{q+1}^{\beta}$$

*for all* $q \geq q_0$. *Then for any* $K > 1$ *and* $\epsilon', \delta' > 0$, *there exists* $h_c, T > 0$, *integer* $q_0 > 0$ *and* $\mu \in (0,1)$ *such that, if* $q \geq q_0$, $\epsilon \in [\varepsilon', K\varepsilon']$, $\delta \in [\delta', K\delta']$ *and* $h \in (0, h_c)$ *then there exists* $\Phi^h \in \Gamma(\varepsilon, \delta)$ *such that (6.22), (6.23) hold for (6.20) so that there exists* $C = C(U_0) > 0$ *such that*

$$dist(U_n, \mathcal{M}) \leq C\mu^n.$$

*Furthermore, there exists* $K > 0$ *such that*

$$\sup_{p \in Y} \|\Phi(p) - \Phi^h(p)\| \leq Kh. \tag{6.24}$$

**Proof** Using (6.20) and Lemma 6.7 it can be shown that, for $h$ sufficiently small, $\tilde{G}$ satisfies the same estimates (6.14) as $G$. (Note that those bounds were doubled to allow incorporation of perturbation error at this stage). Thus existence and attractivity follow as in Theorem 6.4.

The convergence result follows from the uniform contraction principle, Theorem 11.2: note that $\Phi$ and $\Phi^h$ are fixed points of $T$ and $T^h$ respectively, where

$$\begin{aligned} p &= L\xi + \mathbb{P}^m G(\xi + \Psi(\xi)) \\ (T\Psi)(p) &= L\Psi(\xi) + \mathbb{Q}^m G(\xi + \Psi(\xi)), \\ P &= L\xi + \mathbb{P}^m \tilde{G}(\xi + \Psi(\xi)) \\ (T^h\Psi)(P) &= L\Psi(\xi) + \mathbb{Q}^m \tilde{G}(\xi + \Psi(\xi)). \end{aligned} \tag{6.25}$$

Both $\Phi$ and $\Phi^h$ lie in $\Gamma = \Gamma(\varepsilon', \delta')$ and are constructed with contraction constant $\mu$. Thus

$$\|\Phi - \Phi^h\|_\Gamma \leq \frac{1}{1-\mu} \sup_{\Psi \in \Gamma} \|(T\Psi) - (T^h\Psi)\|_\Gamma$$

where

$$\|\Psi\|_\Gamma := \sup_{p \in Y} \|\Psi(p)\|.$$

Now

$$\|T\Psi - T^h\Psi\|_\Gamma = \sup_{p\in Y} \|(T\Psi)(p) - (T^h\Psi)(p)\|.$$

Hence, by (6.25) and Lemma 6.7 we have

$$\begin{aligned}
&\|(T\Psi)(p) - (T^h\Psi)(p)\| \\
&\quad\le \|(T\Psi)(p) - (T^h\Psi)(P)\| + \|(T^h\Psi)(P) - (T^h\Psi)(p)\| \\
&\quad\le \|\mathbb{Q}^m G(\xi + \Psi(\xi)) - \mathbb{Q}^m \tilde{G}(\xi + \Psi(\xi))\| + \delta\|P - p\| \\
&\quad\le (1+\delta)\|G(\xi + \Psi(\xi)) - \tilde{G}(\xi + \Psi(\xi))\| \qquad (6.26) \\
&\quad\le (1+\delta)\|\tilde{E}(\xi + \Psi(\xi), T)\| \\
&\quad\le C(T)(1+\delta)h.
\end{aligned}$$

The convergence result follows. □

An inertial manifold for (6.20) is a set, defined through a Lipschitz graph $\Phi$ and constant $R > 0$, of the form

$$\mathcal{M}^h := \{v \in V : \mathbb{Q}^m v = \Phi^h(\mathbb{P}^m v)\} \bigcap B(0,R). \qquad (6.27)$$

The set is assumed to be positively invariant under (6.20) and to exponentially attract all solutions of (2.9) at a uniform rate; that is, there exists $\nu > 0$ such that, for all $u_0$ in $V$,

$$\exists C = C(u_0) : \mathrm{dist}(S^h(t)u_0, \mathcal{M}^h) \le Ce^{-\nu t} \quad \forall t \ge 0. \qquad (6.28)$$

**Theorem 6.9 (Continuity of Inertial Manifolds)** *Under the assumptions of Theorems 6.4 and 6.8 there exists $h_c > 0$ such that, for $h \in (0, h_c)$, the semigroup $S^h(t)$ has an inertial manifold $\mathcal{M}^h$ satisfying (6.27) and (6.28) for $\Phi^h \in \Gamma(\varepsilon', \delta')$ and $\mu > 0$ given in Theorem 6.8 and $R$ given in (6.27). Furthermore there exists $K > 0$ such that*

$$d_{\mathrm{H}}(\mathcal{M}^h, \mathcal{M}) \le Kh.$$

**Proof** The existence part of the proof may be proved from (6.8) in the same way that Theorem 6.5 follows from Theorem 6.4, using the fact that $\tilde{S}^h(t)$ coincides with the semigroup $S^h(t)$ in the positively invariant set $B(0,R)$. The convergence result follows from (6.24): if $x \in \mathcal{M}^h$ then $x = p + \Phi^h(p)$ for some $p$; set $z = p + \Phi(p)$. Then

$$\mathrm{dist}(\mathcal{M}^h, \mathcal{M}) \le \sup_{x\in\mathcal{M}^h} \|x - z\| \le \sup_{p\in Y} \|\Phi(p) - \Phi^h(p)\| \le Kh$$

as required. Similarly the result follows for $\text{dist}(\mathcal{M}, \mathcal{M}^h)$. □

### 6.4 Bibliography

Inertial manifolds were first introduced by Foias *et al.* [38]. However, the basic idea that certain partial differential equations behave finite-dimensionally for large time can be traced back considerably earlier — see Foias and Prodi [36] and Constantin *et al.* [20], for example. A variety of analytical results concerning inertial manifolds may be found in Constantin *et al.* [21], Foias *et al.* [39], Mallet-Paret and Sell [78] and Chow *et al.* [17]. The original paper of Foias *et al.* [38] employs an existence technique known as the Lyapunov–Perron approach and, using this theory, the effect of perturbation due to a spectral approximation is studied. Demengel and Ghidaglia [22] use similar analytical techniques to study a particular time discretization. Jones and Stuart [65] use a Hadamard graph transform technique (related to that used by Mallet-Paret and Sell [78] in their existence theory) to construct a perturbation theory general enough to allow consideration of a variety of space and time approximations. It is this theory which is outlined here.

It is also worth mentioning that the idea of inertial manifolds has been used in the development of numerical methods, sometimes referred to as nonlinear Galerkin methods, whereby some attempt is made to approximate the inertial manifold by a graph $\Phi_{approx}$ and compute with a spectral method of the form

$$\frac{du^{\mathrm{N}}}{dt} + Au^{\mathrm{N}} = \mathbb{P}F(u^{\mathrm{N}} + \Phi_{approx}(u^{\mathrm{N}})), t > 0, \quad u(0) = \mathbb{P}u_0^{\mathrm{N}}. \qquad (6.29)$$

Such numerical methods are studied in, for example, Devulder *et al.* [24], Foias *et al.* [37], Foias *et al.* [39], Jolly *et al.* [60], Jones *et al.* [63], Russel *et al.* [83][84], Temam [93] and Titi [94]. Note that standard spectral methods correspond to taking $\Phi_{approx} \equiv 0$.

## 7 Attractors

### 7.1 Introduction

In Section 4 we considered the effect of perturbation on a very simple invariant set: the equilibrium point. We also discussed the effect of perturbation on trajectories in the neighborhood of the equilibrium point. Similar analyses can also be carried out for periodic solutions as mentioned in the bibliography of that section. In this section it is our purpose to study the very complicated invariant sets that arise in systems often loosely termed *chaotic*. The notion of an attractor formalizes the concept of a general object which captures the (possibly chaotic) long time dynamics of a system.

Section 7.2 contains the basic definitions of attractors and some of their properties. In particular, Theorem 7.3 gives a useful method for constructing global attractors and Theorem 7.6 is a very useful characterization of the global attractor as the union of all solutions of (2.9) which are defined and bounded for all positive and negative time. Further results of particular importance in Section 7.2 are the characterization of gradient systems and their attractors: see Theorem 7.7 and Corollary 7.8.

The definitions of upper- and lower-semicontinuity from Section 2.3 are required to understand fully the remainder of Section 7. In Section 7.3 we study the upper-semicontinuity of attractors — see Theorem 7.9. This shows that, after a sufficiently long time, every computed point is close to a point on the true attractor. It does not show, however, that every point on the true attractor has a nearby counterpart in the approximate attractor. In other words parts of the attractor may disappear under perturbation. However, if the true and perturbed attractors are uniformly exponentially attracting then the whole attractor perturbs smoothly under the approximation and the attractor is both upper- and lower-semicontinuous — see Theorem 7.10 in Section 7.4. In Section 7.5 we look at another situation where both upper- and lower-semicontinuity can be proved; this arises when the attractor is the union of the closure of unstable manifolds of equilibria. The simplest situation where this arises is for gradient systems, but other possibilities are also included.

### 7.2 Background theory

**Definition 7.1** *A set $A$ attracts a set $B$ under $S(t)$ if, for any $\varepsilon > 0$, there exists $t^* = t^*(\varepsilon, B, A)$ such that $S(t)B \subset \mathcal{N}(A, \epsilon)\ \forall t > t^*$. A compact invariant set $A$ is said to be an attractor if $A$ attracts an open neighborhood of itself. A global attractor is an attractor which attracts every bounded set in $V$.*

**Example 7.2** Consider the equation

$$u_t + Au = \lambda u$$

with the operator $A$ given in Example 2.1 with $\Omega = (0, 1)$. If $\lambda \in (\pi^2, 2\pi^2)$ then 0 attracts the set

$$B = \{v \in V : \int_0^1 v(x)\sin(\pi x)dx = 0\}.$$

However 0 is not an attractor since any open neighborhood contains points $u_0 = a\sin(\pi x)$ for which $\|u(t)\| \propto \exp\{(\lambda - \pi^2)t\}$. If $\lambda \in (0, \pi^2)$ then 0 attracts an open neighborhood of itself and is hence an attractor. It is in fact a global attractor.

Attractors are often constructed by applying the following theorem.

**Theorem 7.3** *Assume that there exists $\tau \geq 0$ and $B \subset V$, a bounded open set, such that $S(t)\bar{B} \subset B\ \forall t \geq \tau$ and $\bigcup_{t \geq \tau} S(t)B$ is relatively compact.*

*Then $\omega(B)$ is an attractor which attracts $B$. Furthermore*

$$\mathcal{A} := \omega(B) = \bigcap_{t\geq 0} S(t)B.$$

**Proof** Since $S(t)\bar{B} \subset B$ for $t \geq \tau$ it follows that

$$\omega(B) = \bigcap_{s\geq\tau} \overline{\bigcup_{t\geq s} S(t)B} \subset \bigcap_{s\geq\tau} \overline{\bigcup_{t\geq s} B} = \bar{B}. \tag{7.1}$$

Thus $\omega(B)$ is bounded. Note that, in fact, $\omega(B)$ is compact and invariant by Theorem 2.11.

We now show that $\mathcal{A} := \omega(B)$ attracts $B$. Assume that it does not. Then there exist $\epsilon > 0$ and sequences $x_k \in B$ and $t_k \to \infty$ such that $S(t_k)x_k \not\subset \mathcal{N}(\mathcal{A}, \epsilon)$. But $S(t_k)x_k$ is a sequence contained in a compact set and hence has a convergent subsequence $S(t_{k_i})x_{k_i} \to y \in \overline{B}$. By Definition 2.10 $y \in \omega(B) = \mathcal{A}$ and this is a contradiction.

Note that $\omega(B) \subset \bar{B}$ by (7.1). We show that, in fact, $\omega(B) \subset B$. Assume for the purposes of contradiction that $\exists y \in \omega(B) \bigcap \partial B$ (where $\partial B = \bar{B}\backslash B$). Since $\omega(B)$ is invariant it follows that, for any $t > 0$ $\exists x \in \omega(B) : S(t)x = y$. But, since $\omega(B) \subset \bar{B}$ we have $x \in \bar{B}$ and hence, by assumption, $y = S(t)x \in B$ for $t \geq \tau$. This is a contradiction and thus no such $y$ exists. Thus $\omega(B) \subset B$.

Now, since $\omega(B) \subset B$ is closed it follows that, for $\varepsilon$ sufficiently small, $\mathcal{N}(\omega(B), \varepsilon) \subset B$. Since $\omega(B)$ attracts $B$ it follows that $\omega(B)$ attracts an open neighborhood of itself and is hence an attractor.

Finally, since $\omega(B) \subset B$ and $\omega(B)$ is invariant we have $\omega(B) \subset S(t)B$ for all $t \geq 0$. Hence

$$\omega(B) \subset \bigcap_{t\geq 0} S(t)B.$$

Furthermore, since

$$S(s)B \subset \overline{\bigcup_{t\geq s} S(t)B}$$

it follows that

$$\bigcap_{s\geq 0} S(s)B \subset \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B} = \omega(B).$$

The final result follows. □

Recall the Definition 2.13 of dissipativity. It follows that a dissipative dynamical system which has some smoothing properties giving compactness will have a global attractor, by Theorem 7.3. Thus we have

**Corollary 7.4** *Assume that $S(t)$ generates a dissipative dynamical system with absorbing set $B$. Then $\omega(B)$ is an attractor which attracts $B$.*

*Furthermore*

$$\mathcal{A} := \omega(B) = \bigcap_{t \geq 0} S(t)B.$$

Another corollary concerns the abstract sectorial evolution equation of Section 2.5.

**Corollary 7.5** *Consider the dynamical system generated by eqn (2.9) under (2.11). There exists $R > 0$ such that $\omega(B(0,R)) = \bigcap_{t \geq 0} S(t)B(0,R)$ is a global attractor for (2.9).*

**Proof** From (2.9), (2.11), the variation of constants formula (2.15) and Lemma 10.6 we have

$$\|u(t)\| \leq e^{-\delta t}\|u(0)\| + \int_0^t \frac{Ce^{-\delta(t-s)}}{(t-s)^{\beta}} ds.$$

Hence

$$\|u(t)\| \leq e^{-\delta t}\|u(0)\| + K \tag{7.2}$$

where

$$K := \int_0^{\infty} \frac{Ce^{-\delta\tau}}{\tau^{\beta}} d\tau.$$

Choosing $R = K + \varepsilon$ and setting $B = B(0,R)$ we deduce from (7.2) that $S(t)\overline{B} \subset B$ for $t \geq t_0$, where $t_0$ is chosen so that

$$e^{-\delta t_0}(K + \varepsilon) < \varepsilon.$$

Furthermore, by applying the compactness estimate of Assumption (2.3), which bounds $|u(t)|_{\eta}$, on the time intervals $[0, 2t_0], [t_0, 3t_0], [2t_0, 4t_0], \ldots$, we deduce that $\bigcup_{t \geq t_0} S(t)B$ is relatively compact. Thus $\mathcal{A}$ attracts $B$ by Theorem 7.3. Furthermore, (7.2) shows that for any bounded set $E$ there exists $T = T(E)$ such that $S(t)E \subset B$ for all $t \geq T$. Hence $\mathcal{A}$ is a global attractor. □

The following characterization of the global attractor is very useful:

**Theorem 7.6** *Consider a dynamical system $S(t)$ with global attractor $\mathcal{A}$. The set $\mathcal{A}$ is equivalent to the union of all complete bounded orbits of $S(t)$.*

**Proof** Let $x \in \mathcal{A}$. Since $\mathcal{A}$ is invariant it follows that $S(t)x \in \mathcal{A}$ and $\exists y \in \mathcal{A} : S(t)y = x$, for every $t > 0$. Thus a complete orbit through $x$ exists and is bounded since $\mathcal{A}$ is compact. This shows that $\mathcal{A}$ is contained in the union of all bounded complete orbits.

Now let $x$ be a point on a complete bounded orbit $H$. Note that $H$ is invariant and hence, for any $t > 0\, \exists y^t \in H : S(t)y^t = x$. We prove that $H \subseteq \mathcal{A}$. Assume that it is not, for contradiction; then, for any $\varepsilon$ sufficiently small, $\exists x \in H : x \notin \mathcal{N}(\mathcal{A}, \varepsilon)$. But, since $\mathcal{A}$ is a global attractor and $H$ is bounded, $\exists t^* > 0$ :

$$x = S(t)y^t \in S(t)H \subseteq \mathcal{N}(\mathcal{A}, \varepsilon) \quad \forall t \geq t^*;$$

thus $x \in \mathcal{N}(\mathcal{A}, \varepsilon)$, a contradiction. This completes the proof. □

The most general results concerning the effect of perturbation on attractors are quite weak and structure must be placed on the attractor to obtain stronger results. In this context a particular class of systems of interest to us are gradient systems. Recall the definition of $\mathcal{E}$ given in (4.1) and the Definition 2.15. Recall also the Definition 5.1 of the unstable set. The following theorem and corollary elucidate the behavior of trajectories and the structure of the attractor for gradient systems.

**Theorem 7.7** *If $S(t)$ defines a gradient system then, for any $u_0 \in V$, $\omega(u_0) \subseteq \mathcal{E}$ and for any negative orbit $\{\varphi(t), t \leq 0\}$ through $u_0$ for which $\bigcup_{t \leq t_1} \varphi(t)$ is relatively compact, $\alpha(u_0) \subseteq \mathcal{E}$. If, in addition, the set $\mathcal{E}$ comprises only isolated points then, for any $u_0 \in V$, there exists $x \in \mathcal{E}$ such that $\omega(u_0) = x$ and, for any negative orbit $\{\varphi(t), t \leq 0\}$ through $u_0$ for which $\bigcup_{t \leq t_1} \varphi(t)$ is relatively compact, there is $y \in \mathcal{E}$ such that $\alpha(u_0) = y$.*

**Proof** Consider the case of $\omega$ limit sets; the argument for $\alpha$ limit sets is similar. As in the proof of Theorem 7.5 we deduce that, for any $\tau > 0$, $\bigcup_{t \geq \tau} S(t)u_0$ is relatively compact. Hence $\omega(u_0)$ is non-empty, compact, invariant and connected by Theorem 2.11. Now let $x$ and $y$ be two points in $\omega(u_0)$ so that there are sequences $t_i, \tau_i \to \infty$ with $t_i < \tau_i < t_{i+1}$ so that $S(t_i)u_0 \to x$ and $S(\tau_i)u_0 \to y$. Since

$$\mathcal{V}(S(t_{i+1})u_0) \leq \mathcal{V}(S(\tau_i)u_0) \leq \mathcal{V}(S(t_i)u_0)$$

and there exists $c > 0$ such that $\mathcal{V}(S(t_i)u_0) \to c$ by continuity of $\mathcal{V}$, we deduce that $\mathcal{V}(S(\tau_i)u_0) \to c$ also. Thus $\mathcal{V}(y) = c$. Since $\omega(u_0)$ is invariant, for each $t \in \mathbb{R}$ we may choose a $y$ such that $y = S(t)x$ — here we have extended $S(\bullet)$ to negative arguments for brevity — and deduce that $\mathcal{V}(S(t)x) = \mathcal{V}(x)$ for all $t \in \mathbb{R}$. By (iv) of Definition 2.15 we have $x \in \mathcal{E}$. By connectedness of the limit set we deduce that $\omega(u_0)$ must be a single point if $\mathcal{E}$ comprises only isolated points. □

**Corollary 7.8** *If $S(t)$ defines a gradient system then it has a global attractor given by*

$$\mathcal{A} = W^u(\mathcal{E}) := \{u_0 \in V : \textit{ a negative orbit } \{\varphi(t), t \leq 0\} \textit{ exists through } u_0 \textit{ and } dist(\varphi(t), \mathcal{E}) \to 0 \textit{ as } t \to -\infty\}.$$

*If, in addition, $\mathcal{E}$ comprises isolated points, then*

$$\mathcal{A} = \bigcup_{v \in \mathcal{E}} W^u(v).$$

**Proof** By Corollary 7.6 the set $\mathcal{A}$ comprises all bounded complete orbits. Since $\mathcal{A}$ is compact by definition, each such complete orbit is compact. Applying Theorem 7.7 the result follows. □

### 7.3 Upper-semicontinuity of attractors

In this section we prove a basic result concerning the upper semicontinuity of attractors. We assume that $S(t)$ and $S^h(t)$ have global attractors. That the perturbation has a global attractor can often be proved directly for many particular approximation schemes. We discuss this briefly in Section 9. However, note that if $S(t)$ has a global attractor, the general Assumptions 3.2 made here would generally only imply the existence of a local attractor; a theory can be developed to cater for this case also.

**Theorem 7.9 (Upper-Semicontinuity of Attractors)** *Consider the approximation of the semigroup $S(t)$ by $S^h(t)$. Assume that $S(t)$ has a global attractor $\mathcal{A}_0$ and assume that there exist $h_0 > 0$ and a bounded $B \subset V$ such that $S^h(t)$ has a global attractor $\mathcal{A}^h$ for each $h \in (0, h_0]$ and*

$$\bigcup_{h\in[0,h_0]} \mathcal{A}^h \subseteq B.$$

*Then*

$$dist(\mathcal{A}_h, \mathcal{A}) \to 0 \quad as \quad h \to 0.$$

**Proof** First note that under Assumptions 3.2 we may assume, without loss of generality, that there exists $t_0 > 0$ and $g(t)$, bounded, continuous and monotonic increasing for $t \geq t_0$, such that

$$\|S^h(t)u - S(t)u\| \leq C(B)g(t)h \quad \forall u \in B, t \geq t_0. \tag{7.3}$$

Furthermore, since $\mathcal{A}_0$ attracts $B$, it follows that there exists a bounded, continuous function $f(t)$, defined for $t > 0$ and decreasing monotonically to zero, such that

$$\text{dist}(S(t)B, \mathcal{A}_0) \leq C(B)f(t) \quad \forall t \geq 0. \tag{7.4}$$

Note also that, since $\mathcal{A}^h$ is invariant under $S^h(t)$ and since $v \in S(t)\mathcal{A}^h$ implies that there exists $w \in \mathcal{A}^h$ such that $v = S(t)w$, we have from (7.3)

$$\begin{aligned}
\operatorname{dist}(S^h(t)\mathcal{A}^h, S(t)\mathcal{A}^h) &= \sup_{u \in S^h(t)\mathcal{A}^h} \left\{ \inf_{v \in S(t)\mathcal{A}^h} \|u - v\| \right\} \\
&= \sup_{u \in \mathcal{A}^h} \left\{ \inf_{v \in S(t)\mathcal{A}^h} \|S^h(t)u - v\| \right\} \\
&= \sup_{u \in \mathcal{A}^h} \left\{ \inf_{w \in \mathcal{A}^h} \|S^h(t)u - S(t)w\| \right\} \\
&\leq \sup_{u \in \mathcal{A}^h} \|S^h(t)u - S(t)u\| \\
&\leq C(B)g(t)h.
\end{aligned} \tag{7.5}$$

Also we have, since $\mathcal{A}^h$ is invariant under $S^h(t)$, since (7.4) holds and since $\mathcal{A}^h \subseteq B$,

$$\begin{aligned}
\operatorname{dist}(\mathcal{A}^h, \mathcal{A}) &\leq \operatorname{dist}(\mathcal{A}^h, S(t)\mathcal{A}^h) + \operatorname{dist}(S(t)\mathcal{A}^h, \mathcal{A}) \\
&\leq \operatorname{dist}(S^h(t)\mathcal{A}^h, S(t)\mathcal{A}^h) + C(B)f(t) \\
&\leq C(B)g(t)h + C(B)f(t).
\end{aligned} \tag{7.6}$$

Thus

$$\operatorname{dist}(\mathcal{A}^h, \mathcal{A}) \leq C(B)[hg(t) + f(t)] \quad \forall t > 0. \tag{7.7}$$

By the properties of $f(\bullet)$ and $g(\bullet)$ we may choose $h_c > 0$ and $t^* = t^*(h)$ such that

$$hg(t^*) = f(t^*) \quad \forall h \in (0, h_c]. \tag{7.8}$$

Figure 2 illustrates this situation. Furthermore, by the monotonicity of $f$ and $g$ it follows that $t^*(h) \to \infty$ as $h \to 0$. Thus, by (7.7),

$$\operatorname{dist}(\mathcal{A}^h, \mathcal{A}) \leq 2C(B)f(t^*(h)). \tag{7.9}$$

By the properties of $f(\bullet)$ and $t^*(\bullet)$ the required result follows. □

### 7.4 Continuity for exponentially attracting attractors

Note that Theorem 7.9 does not necessarily give a rate of convergence for the quantity $\operatorname{dist}(\mathcal{A}_h, \mathcal{A})$ which is a power of $h$. This is since nothing is assumed about the *rate of attraction* of the attractor. If the rate is assumed exponential then a stronger result can be proved and we obtain the error bound given in Theorem 7.10 below. Note that the bound is less than the rate of convergence of individual trajectories, unstable manifolds or inertial

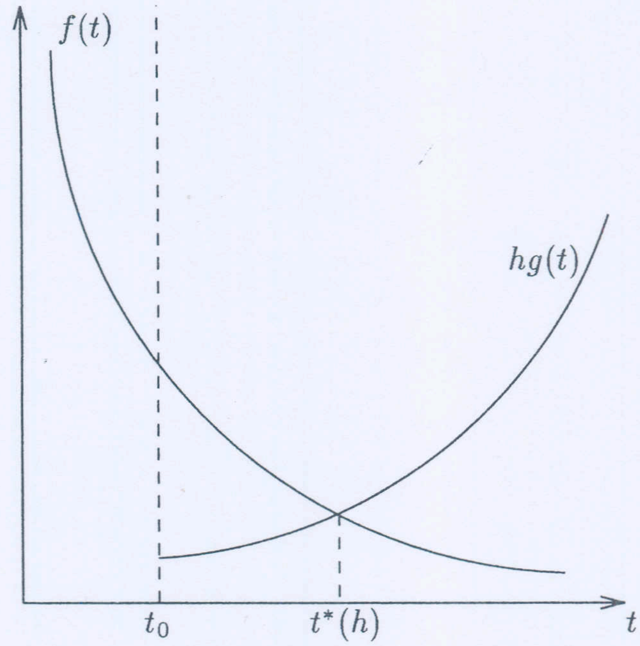


FIG. 2.

manifolds and reflects the competition between the exponential attraction to $\mathcal{A}$ (which determines $\eta$) and the exponential divergence of trajectories on $\mathcal{A}$ (which determines $\alpha$).

**Theorem 7.10 (Continuity of Exponentially Attracting Attractors)** *Consider the approximation of the semigroup $S(t)$ by $S^h(t)$ satisfying (3.2). Let $S(t)$ have a global attractor $\mathcal{A}_0$ and assume that there exist $h_0 > 0$ and a bounded $B \subset V$ such that $S^h(t)$ has a global attractor $\mathcal{A}^h$ for each $h \in (0, h_0]$ and*

$$\bigcup_{h \in [0,h_0]} \mathcal{A}^h \subseteq B.$$

*Assume also that there exists $\alpha, \eta, t_0 \in \mathbb{R}^+$ such that the approximation error satisfies (7.3) with $g(t) = e^{\alpha t}, t \geq t_0$ and that the attractors $\mathcal{A}^h$ are uniformly exponentially attracting so that (7.4) holds with $f(t) = e^{-\eta t}$. Then there exists $K > 0$ such that*

$$d_{\mathrm{H}}(\mathcal{A}_h, \mathcal{A}) \leq K h^{\beta} \quad \forall h \in (0, h_c],$$

*where $\beta = \eta/(\alpha + \eta)$.*

**Proof** Consider $\mathrm{dist}(\mathcal{A}^h, \mathcal{A})$ first. By (7.8) we have

$$he^{\alpha t^*} = e^{-\eta t^*}$$

so that $e^{-t^*} = h^{1/(\alpha+\eta)}$. Hence $f(t^*) = h^\beta$ and the bound follows from (7.9). The bound on $\text{dist}(\mathcal{A}, \mathcal{A}^h)$ follows by reversing the roles of $\mathcal{A}$ and $\mathcal{A}^h$ in the proof of Theorem 7.9. This can be done since the rate of convergence is uniform in $h \in (0, h_c]$. □

### 7.5 Lower-semicontinuity of attractors

The fact that lower-semicontinuity is hard to prove in general is not an artefact of the analysis. Simple examples exist which indicate that lower-semicontinuity is not true in general. Roughly the difficulty is that parts of the attractor which are not exponentially attracting may disappear under perturbation. Unfortunately the uniformly-in-$h$ exponentially attracting attractors of Theorem 7.10 do not arise that often in applications and, even when they do, establishing that they have the right properties can be very hard. Instead we proceed in this section to prove lower-semicontinuity by making assumptions on the nature of the flow on the attractor $\mathcal{A}$. One important case where this is possible is when the dynamical system $S(t)$ is in gradient form and the set $\mathcal{E}$ of equilibria given by (4.1) is a bounded set containing only hyperbolic equilibria. A natural generalization of this is to make the following assumption:

**Assumption 7.11** *The dynamical system (2.9) has a global attractor $\mathcal{A}$ where*

$$\mathcal{A} = \bigcup_{x \in \mathcal{E}} \overline{W^u(x)}$$

*and $\mathcal{E}$ comprises a finite number of hyperbolic equilibrium points of (2.9).*

Note that this assumption is a consequence of the system being in gradient form and having hyperbolic equilibria — see Corollary 7.8; however, Assumption 7.11 is weaker. For example, Assumption 7.11 admits equations with a single unstable equilibrium point and a unique limit cycle attracting all initial data except that starting at the equilibrium point.

Note also that, since the attractor is compact and contains all equilibria and since the hyperbolic fixed points are isolated by Theorem 4.3, the number of fixed points is automatically finite if they are hyperbolic. Under Assumption 7.11 we may prove lower-semicontinuity of the attractor as well as upper-semicontinuity:

**Corollary 7.12 (Lower-Semicontinuity of Attractors)** *Assume that Assumption 7.11 holds and consider the approximation of the semigroup $S(t)$ by $S^h(t)$. Denote the global attractor of $S(t)$ by $\mathcal{A}_0$ and assume that there exist $h_0 > 0$ and a bounded $B \subset V$ such that $S^h(t)$ has a global attractor $\mathcal{A}^h$ for each $h \in (0, h_0]$ and*

$$\bigcup_{h \in [0, h_0]} \mathcal{A}^h \subseteq B.$$

*Then*

$$d_H(\mathcal{A}_h, \mathcal{A}) \to 0 \quad \text{as} \quad h \to 0.$$

**Proof** It follows from Theorem 7.9 that there exists an approximate attractor $\mathcal{A}_h$ satisfying

$$\text{dist}(\mathcal{A}_h, \mathcal{A}) \to 0 \quad \text{as} \quad h \to 0.$$

Thus it remains to establish the lower-semicontinuity result that

$$\text{dist}(\mathcal{A}, \mathcal{A}_h) \to 0 \quad \text{as} \quad h \to 0. \tag{7.10}$$

Let $\bar{u} \in \mathcal{E}$. By Theorem 5.7 we deduce that there exists a fixed point $\bar{u}^h$ of $S^h$ such that

$$\text{dist}(\overline{W^u(\bar{u})}, \overline{W^u_h(\bar{u}^h)}) \to 0 \quad \text{as} \quad h \to 0. \tag{7.11}$$

Now we prove that

$$\overline{W_{\text{h}}{}^u(\bar{u}^h)} \subseteq \mathcal{A}_h. \tag{7.12}$$

Let $x \in W_{\text{h}}{}^u(\bar{u}^h)$; then, by definition, there exists $x_i \to \bar{u}^h$ and $t_i \to \infty$ such that $x = S^h(t_i)x_i$ and hence it follows that $x \in \mathcal{A}^h$ since $\mathcal{A}^h$ is a global attractor and $\{x_i\}_{i=1}^{\infty}$ are contained in a bounded set. Thus (7.11), (7.12) prove that

$$\text{dist}(\overline{W^u(\bar{u}^h)}, \mathcal{A}_h) \to 0 \quad \text{as} \quad h \to 0.$$

Since Assumption 7.11 holds it is clear that (7.10) follows and the proof is complete. □

## 7.6 Bibliography

For background theory on attractors and related material see Babin and Vishik [4], Bhatia and Szego [11], Hale [50] and Temam [92]. In particular the construction of global attractors given in Theorem 7.3 is closely related to the presentation in Temam [92]. Gradient systems are discussed extensively in Hale [50]; aside from their physical significance in problems modeled by dynamic energy minimization (see, for example, Elliott [28]), gradient systems are of fundamental importance in the theory of dynamical systems because of the simple characterization of the attractor given in Corollary 7.8 and the robustness to perturbations which follows from this.

The proof of continuity for exponentially attracting attractors given in Theorem 7.10 is taken from similar results in Babin and Vishik [4] and in Hale *et al.* [51]. The proof of upper-semicontinuity of attractors given in Theorem 7.9 is motivated by the proof in Babin and Vishik [4] concerning exponentially attracting attractors; slightly different proofs of upper semicontinuity are available in, for example, Hale *et al.* [51] and Temam [92]. These two general works generated a number of specific applications such as those studied by Dettori [23] and Shen [86]. Closely related results are proved for uniformly asymptotically stable sets (which are positively invariant sets containing the attractor) in Kloeden and Lorenz [67] [68]. Relationships between the results on attractors and the results on uniformly asymptotically stable sets are given in Hill and Süli [57]. For partial differential equations proof of upper-semicontinuity often requires error bounds for non-smooth initial data; see Elliott and Larsson [30], Larsson [74] and Yin-Yan [99]. This can be overcome in certain cases where the underlying equation, and its approximation, have a strong property known as Gevrey regularity [35] — see Lord and Stuart [77]. For analysis of upper-semicontinuity in the context of time-discrete, multistep methods see Hill and Suli [56].

Explicit and simple examples showing why lower-semicontinuity is not true in general may be found in Humphries *et al.* [59] and in Kapitanskii and Kostin [66]. Note, however, that if the stable and unstable manifolds of the equilibrium points for a gradient system intersect transversally then the attractor is exponentially attracting and Theorem 7.10 may be applied to establish upper- and lower-semicontinuity. The first general proofs of lower-semicontinuity for gradient systems, without requiring the transversal intersection property, appear in Hale and Raugel [52]. Related results may also be found in Babin and Vishik [4] and in Kostin [70]. These approaches assume that the attractor is in gradient form with hyperbolic equilibria. A subsequent generalization may be found in Humphries [58] where the attractor for an ordinary differential equation is assumed to be the union of unstable manifolds of hyperbolic equilibria; this includes the assumption of Hale *et al.* [51] but is more general than it. Results closely related to those of Humphries [58] may be found in Kapitanskii and Kostin [66] and in Humphries *et al.* [59] where partial differential equations are studied. Extensions of this approach to non-hyperbolic equilibrium points may be found in Kostin [71] and in Elliott and Kostin [29].

For estimates of the dimension of the attractor of time-discretized quasi-linear partial differential equations, and comparison to dimension estimates for the underlying partial differential equation, see Eden *et al.* [26].

# 8 Error analysis for gradient systems

## 8.1 Introduction

So far we have derived three basic types of error bound for trajectories. The first appears in Assumptions 3.2 and detailed derivations are given for spectral methods in Theorem 3.6 (and its $C^1$ counterpart Theorem 3.7). The Assumptions 3.2 concern a uniform error bound on compact time intervals disjoint from the origin and bounded sets in $V$ (see the remark following the proof of Theorem 3.6). The second appears in Corollary 4.10 and concerns an error bound uniform in $t \geq 2\tau$ for solutions asymptotic to a stable equilibrium point; this bound is not uniform in a bounded set of initial data essentially because initial data from a bounded set can take arbitrarily long to reach a neighborhood of the equilibrium — see the remark following the proof. The third appears in Theorem 4.18 and Theorem 4.19 and concerns an error bound for trajectories comprising a local phase portrait near an equilibrium point. It is uniform in time and uniform across a sufficiently small neighborhood of an equilibrium point. In this section we put these results together in various ways to derive error bounds for gradient systems and other classes of problems with similar properties.

What we would ideally like is to find an error bound which is uniform in large time and uniform across bounded sets of initial data. In general this does not appear to be possible. If, however, we restrict attention to gradient (and other closely related) systems and weaken the notion of approximation to allow piecewise continuous solutions with a finite number of discontinuities, then the goal is attainable. This we show in the following section — see Theorem 8.8.

### 8.2 The result

We start by making an assumption about solutions of (2.9) which is then shown to be satisfied by a certain class of gradient systems. Recall the set $\mathcal{E}$ defined in (4.1).

**Assumption 8.1** *The dynamical system generated by (2.9) satisfies the following properties*

*(i) the set $\mathcal{E}$ comprises a finite number of hyperbolic equilibria $\{z_i\}_{i=1}^N$ and there is a constant $K_1$ such that, for all sufficiently small $\rho > 0$, there is a bounded open set $\mathcal{U} \supset \mathcal{E}$ such that*

$$\mathcal{U} = \bigcup_{i=1}^{N} Q_i, \quad Q_i \cap Q_j = \emptyset \text{ for } i \neq j, \quad Q_i \subseteq \mathcal{N}(z_i, K_1\rho); \tag{8.1}$$

*(ii) for each bounded set $B \subset V$ there is a time $T$ such that, for any $u_0 \in B$ there is $\tau \in [0, T]$ for which $S(\tau)u_0 \in \mathcal{U}$;*

*(iii) there is a constant $K_2 > 0$ such that, for each $u_0 \in V$ and each $\rho > 0$ sufficiently small, there is a subset of $\mathcal{E}$, relabelled $\{z_i\}_{i=1}^M$, times $\{t_0^+, \{t_i^\pm\}_{i=1}^M\}$ satisfying $t_0^+ = 0 \leq t_1^-$, $t_i^- < t_i^+ < t_{i+1}^-$, $i = 1, \ldots, N-1$, $t_{N-1}^- < t_N^+ = \infty$ and $\varepsilon > 0$ such that*

*(a) $S(t)u_0 \in \mathcal{N}(z_i, K_2\varepsilon)$ for all $t \in (t_i^-, t_i^+)$, $i = 1, \ldots, M$;*
*(b) $\mathcal{N}(z_i, K_2\varepsilon) \bigcap \mathcal{N}(z_j, K_2\varepsilon) = \emptyset$ for $i \neq j$;*
*(c) $S(t)u_0 \notin \mathcal{U}$ $\forall t \in [t_i^+, t_{i+1}^-]$, $i = 0, \ldots, M-1$;*
*(d) $S(t)u_0 \to z_N$ as $t \to \infty$.*

Roughly, this states that the solutions of the dynamical system pass through a finite number of small neighborhoods of equilibria before finally entering and remaining in one such neighborhood for all $t$ sufficiently large.

**Definition 8.2** *The semigroup $S(t)$ generated by (2.9) is said to define a standard gradient system if it is a gradient system and*

*(i) there exists $G \in C(V, \mathbb{R})$ such that, for all functions $w \in C^1(\Lambda, V)$ for some open interval $\Lambda \subset \mathbb{R}$,*

$$\frac{d}{d\lambda}\{G(w(\lambda))\} = \langle \frac{dw}{d\lambda}, F(w)\rangle \quad \forall \lambda \in \Lambda; \tag{8.2}$$

*(ii)* $\mathcal{V}(\varphi) := \frac{1}{2}|\varphi|^2_{\frac{1}{2}} - G(\varphi)$;

*(iii)* $\mathcal{V}(\theta) - \mathcal{V}(\varphi) \le \langle A\theta - F(\theta), \theta - \varphi\rangle + C|\theta - \varphi|^2 \ \ \forall \theta, \varphi \in X^1 = D(A)$.

**Important remark** By taking the inner product of (2.9) with $u_t$ it follows that a standard gradient system satisfies

$$\frac{d}{dt}\{\mathcal{V}(u(t))\} = -|u_t|^2 \tag{8.3}$$

and we will use this equation explicitly in the following. Roughly speaking, the constant $C$ in (iii) is a bound from above on the quadratic form constructed from the second derivative of $\mathcal{V}(\bullet)$ and then (iii) follows from Taylor expansion.

**Example 8.3** Consider the reaction-diffusion equation (2.12) of Example 2.5, under the assumptions (2.13). It is shown to be a gradient system in Example 2.16. Further study of (2.12), (2.13) shows that it is in fact a standard gradient system with

$$G(\varphi) = \int_0^1 h(\varphi(x))dx,$$

where $h$ is given by (2.26). All that is not immediately obvious and remains to be checked is (iii). To establish (iii) note that

$$\begin{aligned}\mathcal{V}(\theta) - \mathcal{V}(\varphi) &= \langle A\theta - F(\theta), \theta - \varphi\rangle - \tfrac{1}{2}\langle A(\theta - \varphi), \theta - \varphi\rangle \\ &+ \langle h(\varphi) - h(\theta), 1\rangle + \langle F(\theta), \theta - \varphi\rangle.\end{aligned}$$

Now

$$\langle h(\varphi) - h(\theta), 1\rangle + \langle F(\theta), \theta - \varphi\rangle =$$

$$\int_0^1 \{h(\varphi(x)) - h(\theta(x)) + f(\theta(x))(\theta(x) - \varphi(x))\}\,dx.$$

But

$$|h(\varphi) - h(\theta) - f(\theta)(\varphi - \theta)| \le \frac{C}{2}|\theta - \varphi|^2 \quad \forall \theta, \varphi \in \mathbb{R}$$

by Taylor expansion, (2.26) and (2.13). Hence, since $A$ is positive definite, the result follows. □

We now show that standard gradient systems satisfy Assumptions 8.1 before moving on to prove appropriate error estimates under those assumptions.

**Theorem 8.4** *Assume that the semigroup generated by (2.9) defines a standard gradient system with a finite number of hyperbolic equilibria and that there exists $K_1 > 0$ such that*

$$\mathcal{U} := \{\eta \in X^1 = D(A) : |A\eta - F(\eta)| < \rho\}$$

*satisfies (8.1) with $\delta = K_1\rho$ for all $\rho$ sufficiently small. Then there exists $K_2 > 0$ such that Assumptions* 8.1 *are satisfied with $\varepsilon = K_2\rho$ for all $\rho$ sufficiently small.*

**Proof** The theorem makes Assumptions 8.1(i) a hypothesis. We turn to (ii). Let $u_0 \in B\backslash\mathcal{U}$, with $B$ bounded. Let

$$\tau = \sup\left\{t \in \mathbb{R}^+ | S(s)u_0 \in B\backslash\mathcal{U}\, \forall s \in [0,t]\right\},$$

and note that $\tau > 0$ by continuity. By (8.3) we have

$$\mathcal{V}(u(s_2)) - \mathcal{V}(u(s_1)) = -\int_{s_1}^{s_2} |u_t(s)|^2 ds. \tag{8.4}$$

Hence, in particular,

$$\mathcal{V}(u(\tau)) - \mathcal{V}(u_0) = -\int_0^{\tau} |u_t(s)|^2 ds \leq -\tau\rho^2.$$

But $\mathcal{V}(u_0)$ is bounded by a constant $K(u_0)$ and $\mathcal{V}(u(\tau)) \geq 0$ since $\mathcal{V} \in C(V, \mathbb{R}^+)$ by definition. Hence we deduce that

$$\tau \leq K(u_0)/\rho^2.$$

Point (ii) follows with

$$T = \frac{1}{\rho^2} \sup_{u_0 \in B} K(u_0).$$

In the following the $K_i$ denote constants independent of $E$ which arise in the course of the analysis. We now establish point (iii) of Assumption 8.1. We assume that there exist times $t_i$ and $u_i := u(t_i), i = 1,2$ such that $u(t) \notin \mathcal{U}$ for all $t \in [t_1, t_2]$ and $u(t_i) \in \partial Q_{i_k}$, the boundary of $Q_{i_k}$, for some integers $i_1, i_2$ between 1 and $N$. We will show first that there exists $K_3 > 0$ such that

$$u(t) \in B(z_l, K_3\rho) \quad \forall t \in [t_1, t_2] \quad \text{if} \quad i_1 = i_2 = l. \tag{8.5}$$

Secondly we will show that there exists $K_4 > 0$ such that

$$\mathcal{V}(z_m) - \mathcal{V}(z_l) \le -K_4\rho \quad \text{if} \quad i_1 = l \ne i_2 = m. \tag{8.6}$$

Let us consider the case (8.5). By (8.4) and Definition 8.2(iii) it follows that, for $t \in [t_1, t_2]$,

$$\begin{aligned}
\textstyle\int_{t_1}^t |u_t(s)|^2 ds &\le \int_{t_1}^{t_2} |u_t(s)|^2 ds \\
&\le \mathcal{V}(u_1) - \mathcal{V}(u_2) \\
&\le |Au_1 - F(u_2)|\,|u_1 - u_2| + C|u_1 - u_2|^2 \\
&\le K_1[1 + CK_1]\rho^2 \\
&\le K_5\rho^2.
\end{aligned}$$

Now we deduce that

$$(t - t_1)\rho^2 \le \int_{t_1}^t |u_t(s)|^2 ds \le K_5\rho^2$$

and hence that $t - t_1 \le K_5$.

From the variation of constants formula (2.15) we have

$$u(t) = e^{-At}u(t_1) + \int_{t_1}^t e^{-A(t-s)}F(u(s))ds,$$

$$z_l = e^{-At}z_l + \int_{t_1}^t e^{-A(t-s)}F(z_l)ds.$$

Using Assumption 2.3, which shows that $F \in C(X^\eta, X)$ for some $\eta < 1$, and the Lemma 10.6, we obtain

$$\|u(t) - z_l\| \le \|u(t_1) - z_l\| + \int_{t_1}^t \frac{K_6\|u(s) - z_l\|}{(t-s)^\eta} ds.$$

Using the facts that $\eta < 1$, $t - t_1 \le K_5$ and $\|u(t_1) - z_l\| \le K_1\rho$ the result (8.5) follows by application of the Gronwall Lemma 10.11.

Now consider (8.6). We have from (8.4) that

$$\mathcal{V}(u_2) - \mathcal{V}(u_1) \le -\int_{t_1}^{t_2} |u_t|^2 ds \le -\rho\int_{t_1}^{t_2} |u_t| ds \le -\rho|u_2 - u_1|.$$

But

$$|u_2 - u_1| \ge |z_m - z_l| - |z_m - u_2| - |z_l - u_1|.$$

Since the equilibria are isolated we have that there exists $\zeta > 0$ such that

$$\min_{i \neq j} |z_l - z_m| > \zeta$$

and so, for sufficiently small $\rho$, since

$$|z_m - u_2|, |z_l - u_1| \leq K_1 \rho, \tag{8.7}$$

it follows that $|u_i - u_j| > \zeta/2$. Hence

$$\mathcal{V}(u_2) - \mathcal{V}(u_1) \leq -\frac{\rho\zeta}{2}$$

and, finally,

$$\mathcal{V}(z_m) - \mathcal{V}(z_l) \leq \mathcal{V}(u_2) - \mathcal{V}(u_1) + \mathcal{V}(z_m) - \mathcal{V}(u_2) + \mathcal{V}(z_l) - \mathcal{V}(u_1).$$

But $\mathcal{V}(z_m) - \mathcal{V}(u_2)$ and $\mathcal{V}(z_l) - \mathcal{V}(u_1)$ are both $\mathcal{O}(\rho^2)$ by Definition 8.2 (ii) and (8.7), so the required result follows.

To complete the proof of (iii) note that (c) follows by Theorem 7.7 since the system is in gradient form. Clearly $S(t)u_0$ passes through a finite number $M \leq N$ of the $Q_i$ and we order these so that $\mathcal{V}(z_i) \leq \mathcal{V}(z_j), 1 \leq j \leq i \leq M$ without loss of generality. Define

$$t_i^- = \inf\{t : u(t) \in Q_i\}, \quad t_i^+ = \sup\{t : u(t) \in Q_i\}.$$

By (8.5) we have $u(t) \in \mathcal{N}(z_i, K_3\rho)$ for all $t \in [t_i^-, t_i^+]$. By (8.6) we deduce that if $I_i = [t_i^-, t_i^+]$ then $I_i \cap I_j = \emptyset$ for $i \neq j$ and the result follows. □

**Example 8.5** We know that the reaction-diffusion equation (2.12) of Example 2.5 subjected to (2.13) yields a standard gradient system. Generically all the equilibria are isolated (see Babin and Vishik [4]) and application of the implicit function theorem in this case shows that the equilibria satisfy the remaining hypothesis of Theorem 8.4.

**Definition 8.6** *The function $\tilde{u}(t)$ is said to be a piecewise continuous solution generated by a dynamical system $S(t)$, if there exist an integer $N$, non-negative numbers $\{T_i\}_{i=0}^N$ and elements $\{U_i\}_{i=0}^{N-1}$ of $V$ such that $0 = T_0 < T_1 < T_2 < \cdots < T_N = \infty$ and for $i = 1, \cdots N$*

$$\tilde{u}(t) = S(t - T_{i-1})U_{i-1}, \, T_{i-1} \leq t < T_i.$$

**Definition 8.7** *A piecewise continuous solution of (2.9) is said to be a combined stabilised trajectory of $S$ if there exists $\rho > 0$ and $\{\bar{u}_j\}_{j=0}^{N-1} \in \mathcal{E}$ such that $B(\bar{u}_i, \rho) \bigcap B(\bar{u}_k, \rho) = \emptyset$ for $i \neq k$, with $U_j \in B(\bar{u}_j; \rho)$ for $j = 0, \cdots N-1$ and $\mathcal{V}(\bar{u}_j) < \mathcal{V}(\bar{u}_{j-1})$ for $j = 1, \cdots N-1$.*

**Theorem 8.8** *Let Assumptions 8.1 be satisfied and let $E \subset V$ be bounded. Then, for any $u_0 \in E$, there exists a constant $C = C(E, \tau)$ and a combined* *stabilised trajectory* $\tilde{u}^h(t)$ *of* $S^h(t)$*, such that for any* $u_0 \in E$

$$\sup_{t \geq \tau} \|S(t)u_0 - \tilde{u}^h(t)\| \leq Ch.$$

**Proof** For simplicity assume that $u_0 \in \mathcal{U}$; the case $u_0 \notin \mathcal{U}$ can be handled similarly. Let

$$I_i = (t_i^-, t_i^+), \quad i = 1, \dots, M$$

and remove all such intervals with $|I_i| \leq T_i^*$ where $T_i^* = T^*(z_i)$ from Theorem 4.13 and Corollary 4.14. Relabel the remaining $\{I_i\}_{i=1}^{M_0}$ where $M_0 \leq M$. Define

$$I_i^* = (t_i^- + T_i^*, t_i^+), i = 1, \dots, M_0, \quad J_i = [t_i^+, t_{i+1}^- + T_{i+1}^*], i = 0, \dots, M_0 - 1.$$

Note that

$$|J_i| \leq T_0 + \sum_{j=1}^{N} (T + T_j^*)$$

since, between intervals $I_i^*$ and $I_{i+1}^*$ the solution can pass through at most $N$ other equilibria and can spend at most $T_j^*$ time units in the neighborhood of each $z_j$. Whilst outside $\mathcal{U}$, Assumption 8.1(ii) shows that the solution can spend at most $T$ time units. Thus we have shown that $|J_i|$ is bounded above in terms of $E$, but independently of the specific choice of $u_0 \in E$. Set

$$T_i = t_i^- + T_i^*, \quad U_i = S^h(T_i^*)u_i^h(0)$$

where $u_i^h(T) = u^h(T)$ given in Theorem 4.18 with $\bar{u} = z_i$ for $i = 1, \dots, M_0 - 1$ and $u_{M_0}^h(T) = u^h(T)$ given in Theorem 4.19 with $\bar{u} = z_{M_0}$.

On the interval $I_i^*$ we apply Theorems 4.18, 4.19 to get the required error bounds and on $J_i$ we apply Assumptions 3.2 together with (2.16) to get the required error bound. The constants in Theorems 4.18, 4.19 are independent of $E$ and depend only upon the equilibria $\{z_i\}_{i=1}^N$. The constants in Assumption 3.2 and (2.16) depend only upon $E$ through the time interval $J_i$ which is bounded above in terms of $E$, independently of the specific choice of $u_0$ in $E$. □

**Important remark** Note that the total number of discontinuities is bounded above by the total number of equilibria. Note also that if the solution $S(t)u_0 \to z_i$ as $t \to \infty$ and does not pass through any $Q_j$, $j \neq i$, then the number of discontinuities is at most one; it will in fact be zero if $z_i$ is stable for then the discontinuity on the boundary of $Q_i$ disappears as Theorem 4.19 becomes equivalent to Theorem 4.8 and the uniformly valid approximate solution has the same initial condition as the underlying solution.

### 8.3 Bibliography

The idea that trajectories of gradient systems can be uniformly approximated in time by a piecewise continuous trajectory with a finite number of discontinuities is contained in the book of Babin and Vishik [4]. Their approach is to make all but the first segment of the piecewise approximation lie on the unstable manifold of an equilibrium point. The price to pay for this is that the order of approximation is $\mathcal{O}(h^{\lambda})$ for some $\lambda \in (0,1)$; however, since the unstable manifolds are finite-dimensional this is a finite-dimensional approximation of an infinite-dimensional dynamical system. In this article we have proved a weaker result, in the sense that the pieces of approximating solution are not finite-dimensional in nature, but a stronger rate of approximation, namely $\mathcal{O}(h)$. The approach taken here to prove the results such as Theorem 8.4 originates in Stuart and Humphries [90] where the effect of error control on gradient dynamical systems in finite dimensions is considered. Similar techniques were then used in Elliott and Stuart [32] to study the viscous Cahn–Hilliard equation (3.10).

## 9 Practical numerical stability

The main purpose of the results in Sections 3–8 in the context of computation is to enable interpretation of data gleaned from long-time simulations. In particular we have shown that certain invariant objects persist under numerical perturbations and we have obtained a variety of error estimates. Amongst other results these enable us to: (i) state with confidence the sense in which computations near an equilibrium point make sense; (ii) state with confidence the sense in which data gleaned from numerical simulations on (possibly chaotic) attractors should be interpreted; (iii) state with confidence the sense in which error bounds for trajectories of gradient systems can be viewed as being uniformly valid in time and across a bounded set of initial data.

However, in the context of numerical approximation, the results described in Sections 3–8 do not distinguish between the relative merits of different approximation methods other than in their rate of convergence. It is of some importance to gain an understanding of which numerical methods work well in practice and the concept of *practical numerical stability* is relevant here. For our purposes we shall take this to mean the construction of schemes which preserve some important features of the underlying semigroup under mild or no restrictions on the mesh parameters.

The first illustration is to consider the equation (2.9) under the assumption that there exist $\alpha, \beta \geq 0$ such that

$$\frac{1}{2}|u|_{\frac{1}{2}}^2 - \langle F(u), u\rangle \geq \beta |u|^2 - \alpha.$$

It then follows from (2.9) that

$$\frac{1}{2}\frac{d}{dt}|u|^2 \leq \alpha - \beta|u|^2$$

and hence that the ball $B = \{u \in X : |u|^2 \leq R\}$ is positively invariant for any $R > \alpha/\beta$. Furthermore any bounded set of initial data is mapped inside $B$ in a finite time so that $B$ is absorbing and the system is dissipative in the sense of Definition 2.13. Such results are crucial stepping-stones to establishing the global bounds on the nonlinearity that we *assume* in Section 2.5 and Section 6, but which must typically be proven a priori for many equations arising in applications. One of the earliest works to look at the preservation of dissipativity in the numerical approximation of equations like (2.9) is Foias *et al.* [34], where finite difference and spectral approximations of the Kuramoto–Sivashinsky equation were studied. Elliott and Stuart [32] address similar issues for a finite difference approximation of a reaction-diffusion equation and temporal discretization by a variety of one-step methods. In the paper of Armero and Simo [3], preservation of dissipativity is studied for the Navier–Stokes equations under finite element time approximation and a variety of one-step temporal approximations. The reviews of Stuart and Humphries [89] and of Humphries *et al.* [59] contain surveys of the literature concerning preservation of dissipativity.

A second illustration of the concept of practical numerical stability is the preservation of the gradient structure of Definition 2.15. An early example containing explicit reference to the importance of retaining the Lyapunov functional under approximation is contained in Elliott [28] where finite element spatial approximation, together with some one-step time approximations of the Cahn–Hilliard equation, are considered. Further studies are contained in Elliott and Stuart [31] where finite difference, one-step and multi-step methods are analysed for a reaction-diffusion equation. See also Stuart and Humphries [89] for a review of this subject.

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## 10 Appendix A — Sectorial evolution equations

In this appendix we outline the basic theory of sectorial evolution equations in a separable Hilbert space $X$ with norm $|\bullet|$. The results are all taken from Henry [54] and Pazy [81] with the exception of Lemma 10.12. Note however that the precise definition of "solution" used here is that given in Miklavčič [79] since that used in Henry [54] does not necessarily yield uniqueness. The results in Henry are not changed in any essential way by this change of definition Hale [50]. We let $A$ denote a linear, densely defined operator in a Hilbert space $X$ with compact inverse and eigenvalue and eigenfunction pairs $\{\lambda_i, \varphi_i\}$ ordered so that

$$\mathrm{Re}\{\lambda_i\} \leq \mathrm{Re}\{\lambda_{i+1}\}.$$

Recall that the *resolvent set* of $A$ is the set of $\lambda$ in the complex plane for which $(A - \lambda I)^{-1}$ is a bounded linear operator in $X$. The norm and inner product on $X$ are denoted by $|\bullet|, \langle\bullet,\bullet\rangle$ respectively.

Definitions 10.1, 10.2 and Lemma 10.3 are Definitions 1.3.1, 1.3.3 and and Theorem 1.3.4 of Henry respectively.

**Definition 10.1** *A linear, closed, densely defined operator $A$ in the Hilbert space $X$ is said to be sectorial if, for some $\varphi \in (0, \pi/2), M \geq 1$ and $a \in \mathbb{R}$, its resolvent set $\mathcal{R}$ satisfies*

$$\mathcal{R} \supseteq \mathcal{S} := \{\lambda | \varphi \leq |arg(\lambda - a)| \leq \pi, \lambda \neq a\}$$

*and, furthermore*

$$|(\lambda I - A)^{-1}| \leq M/|\lambda - a| \quad \forall \lambda \in \mathcal{S}.$$

**Definition 10.2** *An analytic semigroup on a Hilbert space $X$ is a family of continuous linear operators on $X$, $\{T(t)\}_{t\geq 0}$, satisfying*

*(i) $T(0) = I, T(t)T(s) = T(t+s), \quad \forall t, s \geq 0$;*
*(ii) $T(t)x \to x$ as $t \to 0^+$ for each $x \in X$;*
*(iii) $t \mapsto T(t)x$ is real analytic on $0 < t < \infty$ for each $x \in X$.*

*The infinitesimal generator $L$ of $T(t)$ is defined by*

$$Lx = \lim_{t\to 0^+} \left\{ \frac{T(t)x - x}{t} \right\}$$

*with domain $D(L)$ consisting of all $x \in X$ for which this limit exists.*

**Lemma 10.3** *If $A$ is a sectorial operator, then $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$.*

Formally we may think of $T(t) = e^{-At}$. Indeed if $A$ is self-adjoint with respect to the inner product $\langle\bullet,\bullet\rangle$, so that any $v \in X$ can be represented as

$$v = \sum_{j=1}^{\infty} v_j \varphi_j, \quad v_j = \langle v, \varphi_j \rangle, \tag{10.1}$$

then it is appropriate to consider $T(t)$ as being given by

$$T(t)v = e^{-At}v = \sum_{j=1}^{\infty} e^{-\lambda_j t} v_j \varphi_j. \tag{10.2}$$

In the case where $A$ is not sectorial an analogous definition can be made. Henceforth we will use $e^{-At}$ to denote $T(t)$. In the non self-adjoint case the precise definition of $T(t)$ is through a contour integral, evaluation of which coincides with the approach outlined here in the self-adjoint case.

With these definitions we may define fractional powers of the operator $A$; the following definition (from Henry, Definition 1.4.1) can be shown to make sense by use of the properties of sectorial operators; $\Gamma(\bullet)$ denotes the $\Gamma$-function.

**Definition 10.4** *If $A$ is a sectorial operator with $\mathrm{Re}\{\lambda_1\} > 0$ then, for $\alpha > 0$, define the fractional powers of $\mathcal{A}$ by*

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-At} dt,$$

*$A^0 = I$ and $A^{\alpha} = (A^{-\alpha})^{-1}$ with $D(A^{\alpha}) = R(A^{-\alpha})$.*

Returning to the case where $A$ is self-adjoint so that any $v \in X$ can be represented as in (10.1) we find that

$$A^{\alpha}v = \sum_{j=1}^{\infty} \lambda^{\alpha} v_j \varphi_j \ \ \forall \alpha \in \mathbb{R}. \tag{10.3}$$

The following two results follow from Henry, Definition 1.4.7 and Theorems 1.4.2, 1.4.3 and 1.4.8.

**Lemma 10.5** *If $A$ is a sectorial operator, then the space $X^{\alpha} = D(A_1^{\alpha})$ is a Hilbert space with norm $|\bullet|_{\alpha} = |A_1^{\alpha} \bullet|$, where $A_1 = A + aI$ for any $a$ such that $A_1$ has positive eigenvalues. If $A$ has compact resolvent and $\alpha > \beta \geq 0$ then the inclusion $X^{\alpha} \subset X^{\beta}$ is compact.*

**Lemma 10.6** *If $A$ is a sectorial operator then for any $\alpha \leq 0$ there exists $K = K(\alpha) < \infty$ such that*

$$|A^{\alpha}| \leq K.$$

*Furthermore, if $\mathrm{Re}\{\lambda_1\} > \delta > 0$ then, for any $\alpha > 0$ there exists $C = C(\alpha) < \infty$ such that*

$$|e^{-At}|_{\alpha} \leq Ct^{-\alpha}e^{-\delta t} \quad \forall t > 0.$$

We sketch the proof of a related result for the case when $A$ is self-adjoint. From (10.3) and (10.2) we have that

$$A^\alpha e^{-At} v = \sum_{j=1}^{\infty} \lambda_j^\alpha e^{-\lambda_j t} v_j \varphi_j.$$

Hence

$$|e^{-At} v|_\alpha^2 = \sum_{j=1}^{\infty} \lambda_j^{2\alpha} e^{-2\lambda_j t} v_j^2,$$

assuming the normalization $|\varphi_j|^2 = 1\ \forall j$. If $t = 0$ and $\alpha \leq 0$ we have

$$|v|_\alpha \leq \lambda_1^\alpha |v|$$

so that $A^\alpha$ is bounded on $X$.

A simple calculation reveals that

$$\max_{y \geq 0} y^{2\alpha} e^{-2yt} = \alpha^{2\alpha} e^{-2\alpha} / t^{2\alpha}.$$

Hence there exists $C = C(\alpha)$ such that

$$|e^{-At} v|_\alpha \leq C / t^\alpha.$$

It is also of use to note that

$$|e^{-At}| \leq 1 \quad \forall t \geq 0 \tag{10.4}$$

if $A$ is self-adjoint.

Now we consider solving the equation

$$\frac{du}{dt} + Au = f(t, u),\ t > 0, \quad u(0) = u_0 \tag{10.5}$$

Here, for $U$ an open subset of $\mathbb{R}^+ \times X^\beta$, $f : U \mapsto X$ satisfies the following: for every $(t, x) \in U$ there is a neighborhood $V \subset U$ and constants $L \geq 0$, $0 < \theta \leq 1$, such that

$$|f(t_1, x_1) - f(t_2, x_2)| \leq L(|t_1 - t_2|^\theta + |x_1 - x_2|_\beta). \tag{10.6}$$

For simplicity we will denote

$$V \equiv X^\beta \quad \text{and} \quad |\bullet|_\beta \equiv \|\bullet\|.$$

Formally we see then that the equation

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0 \tag{10.7}$$

has solution $u(t) = T(t)u_0 = e^{-At} u_0$. This can be made precise by using the definition of "solution" given in Definition 10.8. With this in mind we will frequently use the variation of constants formula obtained formally from (10.5) by using $e^{-At}$ as an integrating factor to yield the integral equation

$$u(t) = e^{-At}u(0) + \int_0^t e^{-A(t-s)} f(s, u(s))ds. \tag{10.8}$$

In this context we define (see Pazy, Definition 4.2.3):

**Definition 10.7** *A mild local solution of eqn (10.5) is a function $u$ in $C([0,T), X)$ satisfying (10.8). A mild solution of (10.5) is a mild local solution for each $T > 0$.*

Having constructed such solutions it is important to understand when they are classical solutions of the equation. With this in mind we define (see Miklavčič [79] and Hale [50])

**Definition 10.8** *A local solution of (10.5) is a function $u : [0,T) \mapsto V$ such that $u(t) \in C([0,T), X)$, $u_t(t)$ exists in $X$ for $t > 0$, $u(t) \in D(A), t > 0$, $f(\bullet, u(\bullet)) \in C([0,T), X)$ and (10.5) is satisfied on $t \in [0,T)$. A solution of (10.5) is a local solution of (10.5) for each $T > 0$.*

The following theorem is given in Pazy, Theorems 6.3.1 and 6.3.3.

**Theorem 10.9** *Assume that $A$ is sectorial, that $-A$ generates a semigroup $T(t)$ satisfying*

$$|T(t)| \leq M$$

*and that $f(t,x)$ satisfies (10.6) for some $\beta \in [0,1)$. Then for any $u_0 \in V$ there exists $\tau = \tau(u_0)$ and, for each $t^* < \tau$, a constant $C = C(\|u_0\|, t^*)$ such that (10.5) has a solution on $[0,\tau)$ and*

$$|\frac{du}{dt}(t)|_\alpha \leq Ct^{\beta-\alpha-1}, \quad \forall \alpha \in [\beta - 1, 1),\ t \in (0, t^*).$$

*Furthermore, if there exists continuous, non-decreasing, real-valued $k(t)$ such that*

$$|f(t,x)| \leq k(t)(1 + \|x\|) \quad \forall t \geq 0, x \in V, \tag{10.9}$$

*then $\tau = \infty$.*

For convenience we denote the solution operator for the nonlinear problem (10.5) by $S(u_0, t)$ so that $u(t) = S(u_0, t)$. This indicates the dependence of the solution on the time $t$ and initial data $u_0$. The following theorem concerns the regularity of the operator $S(\bullet, \bullet)$. Let $dS(\bullet, t)$ denote the Frećhet derivative of $S(x,t)$ with respect to $x$ and let $df(t, \bullet)$ denote the Frećhet derivative of $f(t,x)$ with respect to $x$. The next result is contained in Henry, Theorem 3.4.4 and Corollary 3.4.5.

**Theorem 10.10** *Suppose that $A$ is a sectorial operator in a Hilbert space $X$ and that (10.6) holds. Suppose further that $f : U \mapsto X$ is $C^r$ with* *derivatives continuous on $U$ uniformly in $t$ for $(t,x)$ in a neighborhood of each point in $U$. Then the map $(x,t) \mapsto S(x,t)$ is $C^r$ for each $t > 0$ on its interval of existence. Furthermore, $v(t) = dS(x,t)\xi$ is a mild solution of the equation*

$$v_t + Av = df(t,u)v, \quad v(0) = \xi.$$

We frequently use the variation of constants formula (10.8), together with Lemma 10.6, to analyse the solution of (10.5) and its approximations. In this context the next lemma, from section 1.2.2 of Henry (see also Elliott and Larsson [30]) is fundamental.

**Lemma 10.11** *Assume that $B, C \geq 0$, $\alpha, \beta \in [0,1)$ and $T \in (0,\infty)$. Then there exists $M = M(B,\alpha,\beta,T) < \infty$ such that for any integrable function $u : [0,T] \mapsto \mathbb{R}$ satisfying*

$$0 \leq u(t) \leq Ct^{-\alpha} + B\int_0^t (t-s)^{-\beta}u(s)ds$$

*for $t$ a.e. in $[0,T)$, we have*

$$0 \leq u(t) \leq CMt^{-\alpha}, \quad t \text{ a.e. in } [0,T].$$

The following specific case of the Gronwall lemma will also be of importance to us; a related result is proved in Henry, Theorem 7.1.1.

**Lemma 10.12** *Assume that $B, C, \gamma > 0$ and $\nu \in (0,1]$. Then there is a constant $K = K(\nu) > 0$ such that for any bounded function $u : [0,\infty) \mapsto \mathbb{R}$ satisfying*

$$0 \leq u(t) \leq Ce^{-\gamma t} + B\int_0^t \frac{e^{-\gamma(t-s)}}{(t-s)^{1-\nu}}u(s)ds,$$

*it follows that*

$$u(t) \leq 2C\exp\{(KB^{1/\nu} - \gamma)t\}.$$

**Proof** By setting

$$q(t) = \exp\{(\gamma - KB^{1/\nu}t\}u(t)/C$$

we see that it is sufficient to prove that

$$\|q\|_\infty := \sup_{t \geq 0} q(t) \leq 2.$$

Now if $\delta KB^{1/\nu} = 1$ then

$$q(t) \leq e^{-t/\delta} + B\int_0^t (t-s)^{-1+\nu}\exp\{-(t-s)/\delta\}q(s)ds.$$

Hence

$$\begin{aligned} q(t) &\leq 1 + B\|q\|_\infty \int_0^t \frac{e^{-u/\delta}}{u^{1-\nu}} du \\ &\leq 1 + B\|q\|_\infty \int_0^\infty \frac{e^{-u/\delta}}{u^{1-\nu}} du \\ &\leq 1 + B\|q\|_\infty \int_0^\infty \frac{e^{-v}}{v^{1-\nu}} dv \\ &= 1 + \frac{\|q\|_\infty}{K^\nu} \int_0^\infty \frac{e^{-v}}{v^{1-\nu}} dv \\ &= 1 + \frac{\|q\|_\infty I}{K^\nu}, \end{aligned}$$

where

$$I = \int_0^\infty \frac{e^{-v}}{v^{1-\nu}} dv.$$

Since $\nu > 0$ we may choose $K$ such that $2I = K^\nu$ to obtain

$$q(t) \leq 1 + \|q\|_\infty / 2.$$

Since this is true for all $t \geq 0$ we have the required result. □

The following three results are useful when the independent variable $u$ is translated to $v = u - \bar{u}$, hence introducing a new linear operator $C$. This occurs, for example, when studying properties of (10.5) in the neighborhood of an equilibrium point. Let $\sigma(A)$ denote the spectrum of an operator $A$. The next result follows from Corollary 1.4.5 and Theorem 1.4.8 of Henry.

**Lemma 10.13** *If $A$ is sectorial with $\mathrm{Re}\{\sigma(A)\} > 0$, and if $C$ is a linear operator with $(C - A)A^{-\alpha}$ bounded for some $\alpha \in [0, 1)$, then $C$ is sectorial. Furthermore $D(C_1^\beta) = D(A^\beta)$ if $c$ is chosen so that $C_1 = C + cI$ has $\mathrm{Re}\{\sigma(C_1)\} > 0$; the norms $|A^\beta \bullet|$ and $|C_1^\beta \bullet|$ are equivalent.*

The following theorem is Theorem 1.5.2 in Henry:

**Theorem 10.14** *Let $C$ be a closed linear operator in $X$ and let $\sigma_1(C)$ denote a bounded spectral set of $C$ and $\sigma_2(C)$ its complement in $\sigma(C) \cup \infty$. Then $X = Y \oplus Z$ where $Y, Z$ are the projections of $X$ associated with the two spectral sets $\sigma_1$ and $\sigma_2$. Furthermore, $Y$ and $Z$ are invariant under $C$.*

**Theorem 10.15** *Let $A$ and $C$ satisfy the conditions of Lemma 10.13 and Theorem 10.14. Let $\sigma_1$ satisfy $-\delta < \mathrm{Re}\{\sigma_1(C)\} < -\gamma < 0$ and $\mathrm{Re}\{\sigma_2(C)\} > \gamma > 0$. Then there exists $K_1 > 0$ such that for all $\alpha \in [0, 1]$*

*and* $\alpha - \beta \in [0,1]$

$$|A^\alpha e^{-Ct} y| \le K_1 e^{\delta t} |y|, \quad \forall y \in Y, t > 0;$$
$$|A^\alpha e^{Ct} y| \le K_1 e^{-\gamma t} |y|, \quad \forall y \in Y, t > 0;$$
$$|A^\alpha e^{-Ct} z| \le K_1 t^{-(\alpha-\beta)} e^{-\gamma t} |A^\beta z|, \quad \forall z \in Z, t > 0.$$

*Finally, there is a constant* $K_2 = K_2(T) > 0$ *such that*

$$|A^\alpha e^{-Ct} v| \le K_2 t^{-(\alpha-\beta)} |A^\beta v|, \quad \forall v \in V,\ t \in (0,T].$$

**Proof** The first two results follow from the fact that $C$ restricted to $Y$ is a bounded linear operator and all norms are equivalent on $Y$; see Henry Theorems 1.5.2 and 1.5.3.

For the third result, let $a$ be such that $C_1 = C + aI$ has spectrum with positive real part. By Henry, Theorem 1.4.4 and 1.5.3, we have for $\alpha \in [0,1]$,

$$|C_1^\alpha e^{-Ct} z| \le \frac{c}{t^\alpha} e^{-\gamma t} |z| \quad \forall z \in Z.$$

Hence, if $0 \le \alpha - \beta \le 1$, then

$$|C_1^{\alpha-\beta} e^{-Ct} C_1^\beta z| \le \frac{c}{t^{\alpha-\beta}} e^{-\gamma t} |C_1^\beta z| \quad \forall z \in Z.$$

Thus

$$|C_1^\alpha e^{-Ct} z| \le \frac{c}{t^{\alpha-\beta}} e^{-\gamma t} |C_1^\beta z| \quad \forall z \in Z.$$

By the norm equivalence of $A$ and $C_1$ given in Henry, Theorem 1.4.8, the third point follows. The final point follows by combining the first and third points and noting that $C_1$ is a bounded operator on $Y$. □

## 11 Appendix B — Contraction principles and Taylor expansions

In this appendix we recall the contraction mapping theorem and two important corollaries. The first result is standard and its proof can be found in numerous texts on analysis.

**Theorem 11.1 (Contraction Mapping Theorem)** *Suppose that* $F : B \mapsto B$ *where* $B$ *is a closed subset of a Banach space* $X$ *with norm* $\| \bullet \|$. *Suppose also that* $F(\bullet)$ *is a contraction on* $B$ *with constant* $\mu < 1$, *so that*

$$\|F(v) - F(w)\| \le \mu \|v - w\|, \quad \forall v, w \in B.$$

*Then there is exactly one point* $u \in B$ *such that* $u = F(u)$.

The next two simple corollaries of the contraction mapping theorem are extremely useful.

**Theorem 11.2 (Uniform Contraction Principle)** *Let* $B$ *be a closed subset of a Banach space* $X$ *with norm* $\|\bullet\|$. *Consider a contraction mapping*

*$F : B \mapsto B$ with contraction constant $\mu < 1$ and fixed point $u$ and a one parameter family of contraction mappings $F_\lambda : B \mapsto B$ with contraction constant $\mu$ and fixed point $u_\lambda$ for $\lambda \in (0, \lambda_c)$. If, for any $\lambda \in (0, \lambda_c)$ there exists $\varepsilon(\lambda) > 0$, such that*

$$\|F_\lambda(v) - F(v)\| \leq \epsilon(\lambda) \quad \forall v \in B$$

*then*

$$\|u_\lambda - u\| \leq \frac{\epsilon(\lambda)}{1-\mu}.$$

**Proof** We have

$$F(u) = u, \quad F_\lambda(u_\lambda) = u_\lambda.$$

Thus, using contractivity and the error bound, we have

$$\begin{aligned} \|u - u_\lambda\| &= \|F(u) - F_\lambda(u_\lambda)\| \\ &\leq \|F(u) - F(u_\lambda)\| + \|F(u_\lambda) - F_\lambda(u_\lambda)\| \qquad (11.1) \\ &\leq \mu\|u - u_\lambda\| + \varepsilon(\lambda). \end{aligned}$$

This gives the desired result. □

**Theorem 11.3 (Commuting Contraction Principle)** *Let $B$ be a closed subset of a Banach space $X$ with norm $\| \bullet \|$. Consider a one parameter family of contraction mappings $F_\lambda : B \mapsto B$ with contraction constant $\mu < 1$ and fixed point $u_\lambda$ for $\lambda \in (\lambda_1, \lambda_2)$. Then, if*

$$F_\lambda \bullet F_\eta = F_\eta \bullet F_\lambda \quad \forall \eta, \lambda \in (0, \lambda_2)$$

*it follows that*

*(i) $u_\lambda$ is independent of $\lambda \in (\lambda_1, \lambda_2)$ and we denote it by $\bar{u}$;*

*(ii) if $F_\lambda \bar{u} \in B$ for $\lambda \in (0, \lambda_0)$ then $\bar{u}$ is a fixed point of $F_\lambda$ for all $\lambda \in (0, \lambda_0)$.*

**Proof** Let

$$F_\eta(u_\eta) = u_\eta, \quad F_\lambda(u_\lambda) = u_\lambda \quad \eta, \lambda \in (\lambda_1, \lambda_2).$$

Then

$$\begin{aligned} (F_\lambda \bullet F_\eta)(u_\eta) &= F_\lambda(u_\eta) \\ \Rightarrow \quad (F_\eta \bullet F_\lambda)(u_\eta) &= F_\lambda(u_\eta) \qquad (11.2) \\ \Rightarrow \quad F_\lambda(u_\eta) &= u_\eta. \end{aligned}$$

Thus $u_\eta = u_\lambda$. This proves (i). We denote the fixed point by $\bar{u}$.

For (ii) let $\eta \in (\lambda_1, \lambda_2)$ and $\lambda \in (0, \lambda_0)$. Then

$$\begin{aligned} F_\eta(\bar{u}) &= \bar{u} \\ \Rightarrow \quad (F_\lambda \bullet F_\eta)(\bar{u}) &= F_\lambda(\bar{u}) \\ \Rightarrow \quad (F_\eta \bullet F_\lambda)(\bar{u}) &= F_\lambda(\bar{u}). \end{aligned} \tag{11.3}$$

Since $F_\lambda(\bar{u}) \in B$ it follows by uniqueness of the fixed point in $B$ that $F_\lambda(\bar{u}) = \bar{u}$ and the result follows. □

Once differentiability of nonlinear operators between Hilbert spaces has been established it is natural to consider Taylor expansions; for a broad introduction to the calculus of nonlinear operators in a Hilbert space see Chow and Hale [16]. The following result may be found in that text. We let $Df^j$ denote the $j^{th}$ Frechet derivative of $f$, a multilinear operator.

**Theorem 11.4 (Taylor Expansions)** *Suppose that $X, Y$ are Hilbert spaces and that $U \subset X$ is an open set. Then, if $f \in C^k(U, Y)$,*

$$f(x+h) = f(x) + Df(x)h + \ldots + \frac{1}{(k-1)!} Df^{k-1}(x)h^{n-1}$$
$$+ \frac{1}{(k-1)!} \int_0^1 (1-s)^{n-1} Df^n(x+sh)h^n ds.$$

## 12 Appendix C — Attractive invariant manifolds

The material in this Appendix is generalized from Jones and Stuart [65]. The generalization is to consider maps which are written as graphs $\Phi \in C(Y, Z)$ over a bounded ball in $Y$ as well as the case where the graph is over the whole of $Y$. The former case is useful for the study of unstable manifolds and the latter for inertial manifolds.

We consider the general question of the existence of attractive invariant manifolds for the map:

$$W_{m+1} = M(W_m), \quad M(W) = LW + N(W). \tag{12.1}$$

(Actually, as will be apparent from the theorem, the manifold may only be locally invariant in the case where the graph of $\Phi$ is defined over a compact ball in the $Y$ coordinate.) We assume that $L : V \mapsto V$ is linear and that $N : V \mapsto V$ is nonlinear. The space $V$ is decomposed into two subspaces $Y$ and $Z$ which are assumed invariant under $L$. Thus

$$V = Y \oplus Z, \quad LY = Y, \quad LZ = Z. \tag{12.2}$$

We let $\mathcal{P}$ and $\mathcal{Q}$ denote the projections of $V$ onto $Y$ and $Z$ respectively and define $p_m = \mathcal{P}W_m, q_m = \mathcal{Q}W_m$; thus $W_m = p_m + q_m$. Throughout the construction of the attractive invariant manifold we make the following assumptions.

**Assumption 12.1** *There exist positive constants $a, b, c, B_1, B_2$ such that*

$$\|Lz\| \le a\|z\| \ \ \forall z \in Z; \tag{G1}$$

$$\forall p \in Y,\ \exists! w \in Y \ s.t.\ Lw = p \quad and \quad b\|y\| \le \|Ly\| \le c\|y\| \ \ \forall y \in Y; \tag{G2}$$

$$\|\mathcal{R}(N(u) - N(v))\| \le B_1\|u - v\|, \quad \|\mathcal{R}N(u)\| \le B_2, \tag{G3}$$

*for all $u, v$ such that $\|\mathcal{P}u\|, \|\mathcal{P}v\| \le r$, $\|\mathcal{Q}u\|, \|\mathcal{Q}v\| \le \gamma$ and where $\mathcal{R}$ equals either $I, \mathcal{P}$ or $\mathcal{Q}$. Furthermore, there exist constants $\alpha \in (0, \infty)$ and $\mu \in (0, 1)$ such that*

$$b^{-1}B_1(1 + \alpha) \le \mu; \tag{C1}$$

$$a\gamma + B_2 \le \gamma; \tag{C2}$$

$$\theta := a\alpha + B_1(1 + \alpha) \le \alpha\varphi, \tag{C3}$$

*where $\varphi := b - B_1(1 + \alpha) > 0$ by (C1);*

$$a + B_1(1 + \alpha) \le \mu. \tag{C4}$$

Under these assumptions we will seek an invariant manifold for (12.1) which is representable as the graph of a function $\Phi$, acting on a subspace of $Y$, and satisfying

$$q_m = \Phi(p_m) \Longleftrightarrow q_{m+1} = \Phi(p_{m+1}) \quad \forall m : \|p_m\|, \|p_{m+1}\| \le r \tag{12.3}$$

and is attractive in the sense that

$$\|q_m - \Phi(p_m)\| \le \mu^m \|q_0 - \Phi(p_0)\| \ \ \forall m : \|p_n\| \le r, \|q_n\| \le \gamma, \ \forall n = 0, \ldots, m. \tag{12.4}$$

Let

$$\bar{Y} = \{p \in Y : \|p\| \le r\}.$$

The appropriate space in which we seek $\Phi$ is now defined:

**Definition 12.2** *Let $\Gamma = \Gamma(\gamma, \alpha)$ denote the closed subset of $C(\bar{Y}, Z)$ satisfying*

$$\|\Psi\|_\Gamma := \sup_{p \in \bar{Y}} \|\Psi(p)\| \le \gamma,$$

$$\|\Psi(p_1) - \Psi(p_2)\| \le \alpha\|p_1 - p_2\| \ \ \forall p_1, p_2 \in \bar{Y}.$$

**Theorem 12.3 (Existence of Attractive Invariant Manifolds)** *Suppose that Assumptions* 12.1 *hold for the mapping (12.1) and that $r \ge (b-1)^{-1}B_2$ if $b > 1$ and $r = \infty$ if $b \le 1$. Then there exists a unique $\Phi \in \Gamma(\alpha, \gamma)$ such that (12.3) and (12.4) hold.*

Throughout the remainder of this section we assume that the conditions of this theorem are satisfied without stating this explicitly in every result.

The proof of the theorem will be given in a series of lemmas. We may write (12.1) as

$$p_{m+1} = Lp_m + \mathcal{P}N(p_m + q_m), \tag{12.5}$$

$$q_{m+1} = Lq_m + \mathcal{Q}N(p_m + q_m). \tag{12.6}$$

The graph $\Phi$ giving the invariant manifold is a fixed point of the operator $T : C(\bar{Y}, Z) \mapsto C(\bar{Y}, Z)$ defined by

$$p \quad = \quad L\xi + \mathcal{P}N(\xi + \Phi(\xi)) \tag{12.7}$$

$$(T\Phi)(p) \quad = \quad L\Phi(\xi) + \mathcal{Q}N(\xi + \Phi(\xi)). \tag{12.8}$$

We employ the contraction mapping theorem to prove the existence of a fixed point of $T$. We first show that the map $T$ is well defined.

**Lemma 12.4** *For any $\Phi \in \Gamma$ and $p \in \bar{Y}$ there exists a unique $\xi \in \bar{Y}$ satisfying (12.7).*

*Proof.* We consider the case $b > 1$ and $r \geq (b-1)^{-1}B_2$. The case $b \leq 1$ is similar. Note that by $(G2)$ $L^{-1}$ exists on $Y$. Thus we may consider the iteration

$$\xi^{k+1} = L^{-1}[p - \mathcal{P}N(\xi^k + \Phi(\xi^k))]. \tag{12.9}$$

If $p \in \bar{Y}$, then this map takes $\xi^k \in \bar{Y}$ into $\xi^{k+1} \in \bar{Y}$ since

$$\|\xi^{k+1}\| \leq b^{-1}r + b^{-1}B_2 \leq b^{-1}r + b^{-1}(b-1)r = r.$$

For any two sequences $\{\xi^k\}, \{\eta^k\}$ generated by (12.9) we have, by (G2), since $\Phi \in \Gamma$,

$$\begin{aligned}\|\xi^{k+1} - \eta^{k+1}\| &\leq b^{-1}B\|\xi^k + \Phi(\xi^k) - \eta^k - \Phi(\eta^k)\| \\ &\leq b^{-1}B(1+\alpha)\|\xi^k - \eta^k\|.\end{aligned}$$

By Condition (C1) the mapping is a contraction and the existence of $\xi$ given any $p \in Y$ follows. □

Thus, by Lemma 12.4, $T\Phi : \bar{Y} \to Z$ is well defined. We now show that $T$ maps $\Gamma(\alpha, \gamma)$ into itself.

**Lemma 12.5** *The mapping $T$ defined by (12.8) satisfies $T : \Gamma \mapsto \Gamma$.*

*Proof.* From (12.8) and (G1), (G3) we have for all $p \in \bar{Y}$ and $\Phi \in \Gamma$

$$\begin{aligned}\|(T\Phi)(p)\| &\leq a\|\Phi(\xi)\| + B_2 \\ &\leq a\gamma + B_2.\end{aligned}$$

Thus we have, by (C2), $\|T\Phi\|_\Gamma \leq \gamma$.

Let $p_1, p_2 \in \bar{Y}$. From Lemma 12.4 there exists $\{\xi_i\}_{i=1}^2$ such that (12.8) is satisfied with $p = \{p_i\}_{i=1}^2$. Subtracting the two equations, we obtain

$$\begin{aligned} \|(T\Phi)(p_1) - (T\Phi)(p_2)\| &\leq a\|\Phi(\xi_1) - \Phi(\xi_2)\| \\ &\qquad + B_1\|(\xi_1 - \xi_2) + \Phi(\xi_1) - \Phi(\xi_2)\| \\ &\leq [a\alpha + B_1(1+\alpha)]\|\xi_1 - \xi_2\| \\ &= \theta\|\xi_1 - \xi_2\|, \end{aligned}$$

where we have used $(G1), (G3)$, $(C3)$ and the properties of $\Phi \in \Gamma$. From $(G2)$, $(G3)$ and (12.7) we have

$$\begin{aligned} b\|\xi_1 - \xi_2\| &\leq \|L(\xi_1 - \xi_2)\| \\ &\leq \|p_1 - p_2\| + B_1(1+\alpha)\|\xi_1 - \xi_2\|. \end{aligned}$$

Using $(C1)$ we deduce that $\varphi > 0$, and hence

$$\|\xi_1 - \xi_2\| \leq \frac{1}{\varphi}\|p_1 - p_2\|.$$

Thus $(C3)$ implies

$$\|(T\Phi)(p_1) - (T\Phi)(p_2)\| \leq \frac{\theta}{\varphi}|p_1 - p_2\| \leq \alpha\|p_1 - p_2\|.$$

Hence $T : \Gamma \mapsto \Gamma$. □

Now we may show that the map $T$ is a contraction on the space $\Gamma$.

**Lemma 12.6** *For any $\Phi_1, \Phi_2 \in \Gamma$ we have*

$$\|T\Phi_1 - T\Phi_2\|_\Gamma \leq \mu\|\Phi_1 - \Phi_2\|_\Gamma.$$

*Proof.* By Lemma 2.3, for any $p \in \bar{Y}$ and $\{\Phi_i\}_{i=1}^2 \in \Gamma$ we can find $\{\xi_i\}_{i=1}^2$ such that for $i = 1, 2$

$$p = \mathcal{P}M(\xi_i + \Phi_i(\xi_i)) \qquad (12.10)$$

$$(T\Phi_i)(p) = \mathcal{Q}M(\xi_i + \Phi_i(\xi_i)).$$

Using $(G1), (G3)$ we have

$$\|(T\Phi_1)(p) - (T\Phi_2)(p)\| \leq (a + B_1)\|\Phi_1(\xi_1) - \Phi_2(\xi_2)\| + B_1\|\xi_1 - \xi_2\|;$$

adding and subtracting $\Phi_2(\xi_1)$, using the triangle inequality and $(C3)$, we majorize the last inequality by

$$\|(T\Phi_1)(p) - (T\Phi_2)(p)\| \leq (a + B_1)\|\Phi_1(\xi_1) - \Phi_2(\xi_1)\| + \theta\|\xi_1 - \xi_2\|. \qquad (12.11)$$

Now, using $(G2)$ and (12.10), we have similarly that

$$\begin{aligned} b\|\xi_1 - \xi_2\| &\leq \|L(\xi_1 - \xi_2)\| \leq B_1\|\xi_1 - \xi_2 + \Phi_1(\xi_1) - \Phi_2(\xi_2)\| \\ &\leq B_1(1+\alpha)\|\xi_1 - \xi_2\| + B_1\|\Phi_1(\xi_1) - \Phi_2(\xi_1)\|. \end{aligned}$$

Thus since $\varphi > 0$, by $(C1)$

$$\|\xi_1 - \xi_2\| \leq \frac{B_1}{\varphi}\|\Phi_1(\xi_1) - \Phi_2(\xi_1)\|.$$

Returning to (12.11) and using $(C3), (C4)$, we find

$$\|(T\Phi_1)(p) - (T\Phi_2)(p)\| \leq \mu\|\Phi_1(\xi_1) - \Phi_2(\xi_1)\|. \tag{12.12}$$

The result follows after taking the supremum over $\xi_1 \in \bar{Y}$ and then $p \in \bar{Y}$. □

**Proof of Theorem 12.3** The existence of the manifold $\Phi$ follows from Lemmas 12.4–12.6. To establish the exponential attraction of solutions to this manifold let $W_m = p_m + q_m$ be an arbitrary trajectory of (12.5), (12.6) with $\|p_m\| \leq r, \|q_m\| \leq \gamma$. Set

$$\begin{aligned} p &= Lp_m + \mathcal{P}N(p_m + \Phi(p_m)), \\ \Phi(p) &= L\Phi(p_m) + \mathcal{Q}N(p_m + \Phi(p_m)). \end{aligned}$$

Then using $(G1), (G2)$ we have

$$\begin{aligned} \|q_{m+1} - \Phi(p_{m+1})\| &\leq \|q_{m+1} - \Phi(p)\| + \|\Phi(p) - \Phi(p_{m+1})\| \\ &\leq (a + B_1)\|q_m - \Phi(p_m)\| + \alpha\|p - p_{m+1}\|. \end{aligned}$$

However, by (12.5), we have

$$\|p - p_{m+1}\| \leq B_1\|q_m - \Phi(p_m)\|.$$

Thus by $(C4)$ we obtain

$$\|q_{m+1} - \Phi(p_{m+1})\| \leq \mu\|q_m - \Phi(p_m)\|$$

and the result follows. □