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WHITE NOISE LIMITS FOR INERTIAL PARTICLES IN A RANDOM FIELD*

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Abstract. In this paper we present a rigorous analysis of a scaling limit related to the motion of an inertial particle in a Gaussian random field. The mathematical model comprises Stokes's law for the particle motion and an infinite dimensional Ornstein–Uhlenbeck process for the fluid velocity field. The scaling limit studied leads to a white noise limit for the fluid velocity, which balances particle inertia and the friction term. Strong convergence methods are used to justify the limiting equations. The rigorously derived limiting equations are of physical interest for the concrete problem under investigation and facilitate the study of two-point motions in the white noise limit. Furthermore, the methodology developed may also prove useful in the study of various other asymptotic problems for stochastic differential equations in infinite dimensions.

Key words. inertial particles, Ornstein–Uhlenbeck process, random velocity field, white noise limits

AMS subject classifications. 60H15, 60H30, 60G15, 37H10

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1. Introduction.

Many problems in the sciences and engineering involve the interaction of particles with a field (continuum). Examples include cell-migration in a chemical field [21], neuron modeling [28], and the modeling of spray combusters [1]. It is often the case that the field is active at a broad range of length and time scales and that it is natural to model it by an appropriate stochastic process. Furthermore, if the random field decorrelates rapidly when compared with particle time scales, then it is natural to seek a reduced description of the particle motions in which the effect of the field is replaced by white noise.

For finite dimensional problems containing two widely different time scales, the derivation of effective stochastic differential equations (SDEs) for the slow variables has been thoroughly studied, either through techniques of weak convergence (e.g., [14, 22]) or strong convergence [6]. Recently, a new methodological framework for the study of such problems has been developed with application to problems in the atmospheric sciences [17] and the modeling of membranes immersed in a fluid [12, 13]. These last two examples are notable in that they are infinite dimensional in character. However, although the formalism developed in [14] is used, there are currently no infinite dimensional analogues of the weak convergence theorems which underpin the asymptotic approach used in [12, 13, 17]. On the other hand, when the field is described by a rapidly decorrelating Ornstein–Uhlenbeck (OU) process, strong convergence techniques can be used to rigorously justify elimination of the field to produce white noise effects on the particle motion. This idea is developed by Dowell [3] for a special class of OU processes on manifolds. In this paper we use Dowell's approach to tackle a concrete problem in which the underlying infinite dimensional OU process is quite general. In so doing we describe an approach to the rigorous justification of the elimination of fast scales in stochastic differential equations in infinite dimensions

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(SPDEs) which may be useful in a variety of applications. In addition we provide an instance of dimension reduction for our concrete problem. We generalize Dowell's analysis in two primary directions. First, the contraction semigroup generating the noise is a quite general OU process. Second, we prove mean square convergence in the space of continuous functions, while Dowell proves mean square convergence for each fixed time.

The concrete problem we study is the motion of an inertial particle in a turbulent velocity field. The particle moves on the two-dimensional (2D) unit torus $\mathbb{T}^2$ according to Stokes's law in a 2D incompressible random velocity field. We consider the mathematical model that was introduced in [27] and analyzed in [26]. We look at certain scalings of time and nondimensional parameters in which the field is rapidly decorrelating. By applying the methodology in [12, 13, 17] SDEs were derived which describe particle motions in this asymptotic limit of rapid decorrelation in [27]. Here we give rigorous justification of this procedure.

One of the important applications of the model studied here is the analysis of N-point motions of inertial particles. Thus we are interested in the stochastic flow. Although we do not prove it, the techniques of Dowell [3] can also be used to prove convergence to the limiting SDEs as a stochastic flow, justifying the use of the SDEs to study N-point motions in the relevant asymptotic limits. In this context it is relevant to mention the work of Kesten and Papanicolaou [9, 8]. They study two-point motions in a rapidly spatially decorrelating field, with time decorrelation introduced through Lagrangian motion. The N-point motions for passive tracers moving in a Gaussian, homogeneous, mean zero random field which is delta-correlated in time (GRDT model) have also been studied [16, sect. 4]. Closed equations for the N-particle passive scalar correlation functions have been obtained, and it has been shown that the passive tracer particles move according to coupled Brownian motions. In a recent work Kramer [11] rigorously demonstrated that, when thinking of the GRDT model as the limit of velocity fields with short correlation time, care has to be taken as to how the limit is taken: only under the diffusive rescaling is the limiting behavior of the passive tracer particles adequately described by the GRDT model. Our model corresponds to a diffusive rescaling for the inertial particles, and the stochastic flow generated by the limiting SDEs describes the N-point motions.

1.1. The model. The model for the motion of an inertial particle is described by the following system of equations in nondimensional form:

$$\tau\ddot{x}(t) = v(x(t),t) - \dot{x}, \tag{1.1a}$$

$$v(x,t) = \nabla^{\perp}\psi(x,t), \tag{1.1b}$$

$$\frac{\partial\psi}{\partial t} = -\nu A\,\psi + \sqrt{\nu}\frac{\partial W}{\partial t}, \tag{1.1c}$$

$$W(x,t) = \sum_{k\in K}\sqrt{\lambda_k}\,e_k(x)\,\beta_k(t), \tag{1.1d}$$

where $(x,y) \in \mathbb{T}^2 \times \mathbb{R}^2$, $K = 2\pi\mathbb{Z}^2 \setminus \{(0,0)\}$, and the dots denote differentiation with respect to time t. Moreover, $\nabla^{\perp}$ denotes the skew gradient: $\nabla^{\perp} = (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. The set $\{\beta_k\}_{k\in K}$ comprises standard complex valued Brownian motions: $\{Re\,\beta_k\}_{k\in K}$

and $\{Im\,\beta_k\}_{k\in K}$ are families of independently and identically distributed real valued Brownian motions with variance $\frac{1}{2}$. They are independent for different indices, except for the condition $\beta_k = \bar{\beta}_{-k}$.[1] Further, we let $e_k(x) = e^{ik\cdot x}$. Note that the condition $\beta_k = \bar{\beta}_{-k}$ ensures that the Wiener process $W(x,t)$ is real valued. The spectrum of the Wiener process $\{\lambda_k\}_{k\in K}$ is normalized by [27, sect. 2]

$$\sum_{k\in K} \lambda_k = 1. \tag{1.2}$$

The shape of the spectrum of the Wiener process is specified by a function ζ:

$$\lambda_k = \zeta(|k|), \tag{1.3}$$

where $\zeta(z) = \ell^2\zeta_0(\ell z)$ for some appropriately normalized function $\zeta_0\colon \mathbb{R}^+ \to \mathbb{R}^+$. Notice that if ζ_0 has a single maximum, then the spectrum is maximized for $|k|$ of order $O(1/\ell)$. The nondimensional parameter ℓ is the correlation length. Moreover, ν is the inverse of the correlation time of the velocity field and τ is the time-scale ratio of the particle to the fluid.

We take A to be a positive self-adjoint operator on the space $H = \{f \in L^2_{per}(\mathbb{T}^2); \langle f\rangle = 0\}$ with domain of definition $D(A) \subset H$.[2] In the subsequent analysis we shall assume that the operator A is a diagonal operator $\mathrm{diag}\{\alpha_k\}_{k\in K}$ in the basis $\{e_k(x)\}_{k\in K}$. We will also assume that

$$\alpha_k = \xi(|k|). \tag{1.4}$$

The fact that A is a positive operator implies that all of the diagonal entries are positive. For example, A can be $-\Delta$. In this case $D(A) = \dot{H}_{per}(\mathbb{T}^2) := \{f \in H^2(\mathbb{T}^2); \langle f\rangle = 0\}$, which is dense in H, and $e_k(x)$ are the eigenfunctions of A with corresponding eigenvalues $\alpha_k = |k|^2$. In the subsequent analysis we shall consider problem (1.1) for a general operator A with particular emphasis on the case $A = -\Delta$. Physically, the function ξ determines the relative rates of decorrelation of structures at different length scales. Empirically, it is reasonable to assume that $\xi(z) \to \infty$ as $z \to \infty$, and we make this assumption here. This simplifies the statement of the conditions on the spectrum of the Wiener process but is not necessary from a mathematical point of view. We remark that in Dowell's analysis [3] only the case where A is the identity operator is considered.

Finally, the system of equations (1.1) is augmented with initial conditions $\{x_0, y_0, \psi_0\}$. In this paper we shall assume that x_0, y_0 are random variables with finite dimensional moments (precise conditions will be given in the theorems below) and that ψ_0 is statistically stationary in a sense to be described precisely below.

There are three nondimensional numbers in the system of equations (1.1), namely the inverse correlation time ν, the time-scale ratio τ, and the correlation length ℓ. For time scales of order one the particle distributions generated by (1.1) are well understood [26]. Our goal is to study the large-time behavior of the inertial particles system under appropriate scalings of the correlation time ν^{-1} and the time-scale ratio τ, while keeping the correlation length ℓ fixed. To this end, we set

$$t = s\,\gamma, \quad \tau = \tau_0\,\gamma^{\alpha}, \quad \nu = \nu_0\,\gamma^{\beta} \tag{1.5}$$

[1] We will use the notation $\bar{b}$ to denote the complex conjugate of a number b and the notation b^* to denote the conjugate transpose of a vector or matrix b, or the adjoint of an operator b.

[2] We use the notation $\langle f\rangle = \int_{\mathbb{T}^2} f(x)\,dx$.

and study the regime $\gamma \gg 1$.[3] Both τ_0 and ν_0 are $O(1)$ numbers. It is clear from (1.5) that the exponent α controls the time-scale ratio, whereas the exponent β controls the fluid correlation time. Let also $v_0(x,s)$ be the velocity field which is obtained from (1.1c) for $\nu = 1$. In the following we write $W(\cdot,x)$ and $\beta_k(t)$ for Wiener processes equivalent in law to the previous occurrences with the same notation. We repeatedly use the equivalence in law of $B(c\,t)$ and $c^{1/2}B(t)$ for Brownian motion [7, Chap. 2, Lem. 9.4, par. 9.4]. We substitute the rescalings (1.5) in (1.1) to obtain

$$\tau_0\gamma^{\alpha-2}\frac{d^2\,x}{d\,s^2} = v_0(x,\nu_0\,\gamma^{1+\beta}\,s) - \gamma^{-1}\frac{d\,x}{d\,s}, \tag{1.6a}$$

$$v_0(x,s) = \nabla^{\perp}\psi_0(x,s), \tag{1.6b}$$

$$\frac{\partial\psi_0}{\partial s} = -A\,\psi_0 + \frac{\partial W}{\partial s}, \tag{1.6c}$$

$$W(x,s) = \sum_{k\in K}\sqrt{\lambda_k}\,e_k(x)\,\beta_k(s). \tag{1.6d}$$

In the following we replace s by t, the conventional time variable. We multiply (1.6a) through by $\gamma^{2-\alpha}$ and again use the scaling properties of Brownian motion to rewrite (1.6) in the following form:

$$\tau_0\frac{d^2\,x}{d\,t^2} = \gamma^{2-\alpha}v(x,t) - \gamma^{1-\alpha}\frac{d\,x}{d\,t}, \tag{1.7a}$$

$$v(x,t) = \nabla^{\perp}\psi(x,t), \tag{1.7b}$$

$$\frac{\partial\psi}{\partial t} = -\nu_0\gamma^{1+\beta}A\,\psi + \sqrt{\nu_0}\gamma^{\frac{1+\beta}{2}}\frac{\partial W}{\partial t}, \tag{1.7c}$$

$$W(x,t) = \sum_{k\in K}\sqrt{\lambda_k}\,e_k(x)\,\beta_k(t). \tag{1.7d}$$

Depending upon the specific values of α and β there are three different distinguished limits which lead to a nontrivial white noise effect. In this paper we will study the distinguished limit resulting from the choice $\alpha = \beta = 1$, with $\gamma = \epsilon^{-1}$, $\epsilon \ll 1$. The other two cases will be briefly discussed in section 6.

For the choice of $\alpha = \beta = 1$ the system of equations (1.7) becomes

$$\tau_0\ddot{x} = \frac{1}{\epsilon}v(x,t) - \dot{x}, \tag{1.8a}$$

$$v(x,t) = \nabla^{\perp}\psi(x,t), \tag{1.8b}$$

[3] From a physicist's point of view it would be more natural to write the rescaling in the form $\tau = \tau_0\,\gamma$, $t = s\,\gamma^{\alpha}$, $\nu = \nu_0\,\gamma^{\beta}$, that is, to think of the appropriate time scale at which we get a nontrivial limit, given an assumption about the size of τ. Of course, the two rescalings are mathematically equivalent.

$$\frac{\partial \psi}{\partial t} = -\nu_0 \frac{1}{\epsilon^2} A\,\psi + \sqrt{\nu_0}\frac{1}{\epsilon}\frac{\partial W}{\partial t}, \tag{1.8c}$$

$$W(x,t) = \sum_{k\in K} \sqrt{\lambda_k}\, e_k(x)\, \beta_k(t). \tag{1.8d}$$

In equations (1.8) the velocity field amplitude is the square root of its time scale, which formally leads to a white noise as $\epsilon \to 0$. Moreover, it is evident from this equation that both friction and inertia balance the white noise as $\epsilon \to 0$.

The goal of this paper is to analyze the limiting behavior of the equations of motion (1.8a) as $\epsilon \to 0$. To simplify the notation we set $\nu_0 = \tau_0 = 1$. We prove strong convergence of the solution of (1.8) to the solution of a limiting SDE. Note, however, that equations (1.8) are related to (1.1) only weakly, through the rescaling (1.5).

The solution of the SPDE (1.8c) can be written in the following form:

$$\psi(x,t) = \sum_{k\in K} e_k(x)\, \eta_k(t),$$

where η_k are complex valued OU processes satisfying the reality condition $\eta_k = \bar{\eta}_{-k}$, otherwise independent, and solving

$$d\eta_k = -\frac{1}{\epsilon^2}\,\alpha_k\,\eta_k\,dt + \frac{1}{\epsilon}\,\sqrt{\lambda_k}\,d\beta_k, \quad k \in K. \tag{1.9}$$

We assume that the initial conditions for each mode are mutually independent and statistically stationary, i.e., that $\eta_k(0) \in \mathcal{N}(0, \frac{\lambda_k}{2\alpha_k})$. This choice makes ψ_0, and hence v_0, statistically stationary (except for the reality condition).

It will be more convenient for the subsequent analysis to write (1.8) as a first order system with the equation for the stream function being written in Fourier space:

$$d\,x = y\,dt, \tag{1.10a}$$

$$d\,y = \left(\frac{f(x)\,\eta}{\epsilon} - y\right) dt, \tag{1.10b}$$

$$d\eta = -\frac{1}{\epsilon^2} A\,\eta dt + \frac{1}{\epsilon} dW. \tag{1.10c}$$

Here $\eta := \{\eta_k\}_{k\in K} \in \widehat{\mathbb{C}}^K := \{\eta \in \mathbb{C}^K\,;\, \eta_k = \bar{\eta}_{-k}\}$. We use the notation $\mathbb{C}^K$ to denote the complex Hilbert space of square summable sequences on the 2D lattice K equipped with the ℓ^2 inner product and norm: $(\zeta,\xi)_{\ell^2} = \sum_{k\in K} \zeta_k \bar{\xi}_k$ and $\|\zeta\|_{\ell^2}^2 = \sum_{k\in K} |\zeta_k|^2 < \infty$, respectively. We now redefine the operator $A\colon \widehat{\mathbb{C}}^K \to \widehat{\mathbb{C}}^K$ by $A\gamma = \sum_{k\in K} \alpha_k\,\gamma_k$, $\gamma \in \widehat{\mathbb{C}}^K$. The operator $f(x)\colon \widehat{\mathbb{C}}^K \to \mathbb{R}^2$, for fixed $x \in \mathbb{T}^2$, is defined as

$$f(x)\gamma = \sum_{k\in K} \nabla^\perp e_k(x)\gamma_k, \quad \gamma \in \widehat{\mathbb{C}}^K. \tag{1.11}$$

Finally, the Wiener process $W(t)$ has the following Fourier representation:

$$W(t) = \sum_{k\in K} \sqrt{\lambda_k}\,\hat{e}_k\,\beta_k(t), \tag{1.12}$$

where $\{\hat{e}_k\}_{k\in K}$ is the standard basis in $\mathbb{C}^K$, $\hat{e}_k = [0,\dots,1,\dots]^T$—the eigenfunctions of the operator A. Note that (1.3) and (1.4) imply that $\lambda_k = \lambda_{-k}$ and that $\alpha_k = \alpha_{-k}$. We will not use (1.3) directly, but we do assume the form (1.4), with $\xi(z) \to \infty$ as $z \to \infty$, in order to simplify the spectral conditions arising. In any case, (1.3), (1.4) are natural from an applied perspective.

1.2. Statement of main results. Our goal is to show that the solutions $x(t), y(t)$ of (1.10) converge strongly, as $\epsilon \to 0$, to the solutions $\{X(t),\, Y(t)\}$ of the limiting SDE:

$$dX = Y\,dt, \tag{1.13a}$$

$$dY = f(X)A^{-1}dW - Y dt, \tag{1.13b}$$

with the same initial conditions $\{x_0, y_0\}$. Formally, this equation is derived by noting that (1.10c) gives

$$\epsilon^{-1}\eta dt = A^{-1}dW - \epsilon d\eta,$$

substituting this in (1.10b) and setting $\epsilon = 0$. The limiting SDE is well posed. In fact, we have the following theorem.

THEOREM 1.1. *Consider the limiting SDE* (1.13) *with* $t \in [0,T]$. *Assume that the initial conditions are random variables satisfying* $\mathbb{E}\left(|x_0|^2 + |y_0|^2\right) < \infty$ *and are independent of the* σ*-algebra generated by the Wiener process* $W(t)$. *Assume further that the spectrum of the Wiener process* $W(t)$ *satisfies*

$$\sum_{k\in K} \frac{|k|^4\lambda_k}{\alpha_k^2} < \infty.$$

Then the SDE (1.13) *has a unique* t*-continuous solution* $\{X(t), Y(t)\}$ *which is adapted to the filtration generated by the initial conditions and the Wiener process and satisfies* $\mathbb{E}\left[|X(t)|^2 + |Y(t)|^2\right] < \infty$.

In order to prove convergence of the solutions of (1.10) to the solutions of (1.13) we will need to obtain various estimates on functionals of $x(t)$ and $y(t)$ which are valid uniformly in ϵ. These estimates will lead to various conditions on the spectrum of the Wiener process. For the convenience of the reader we summarize these conditions below:

$$\sum_{k\in K} (\alpha_k)^\delta\, \lambda_k < \infty \ \text{ for some } \delta > 0, \tag{1.14a}$$

$$\sum_{k\in K} \frac{|k|^4\lambda_k}{\alpha_k^2} < \infty, \tag{1.14b}$$

$$\sum_{k\in K} \frac{|k|^4}{\alpha_k^{3+\delta}} < \infty \ \text{ for some } \delta > 0, \tag{1.14c}$$

$$\sum_{k\in K} \frac{|k|^2\lambda_k}{\alpha_k^3} \log(\alpha_k) < \infty, \tag{1.14d}$$

$$\sum_{k\in K} |k|^{4+\epsilon}\frac{\lambda_k}{\alpha_k} < \infty \text{ for some } \epsilon > 0. \tag{1.14e}$$

For the particularly interesting case where $-A$ is the Laplacian we have $\alpha_k = |k|^2$ and conditions (1.14b), (1.14c), and (1.14d) are automatically satisfied on account of the normalization condition (1.2) and the fact that we are in the 2D lattice. Among the remaining two conditions, (1.14e) is clearly more restrictive and becomes

$$\sum_{k\in K} |k|^{2+\epsilon}\lambda_k < \infty \text{ for some } \epsilon > 0. \tag{1.15}$$

It is easy to check that this condition is satisfied for the Kraichnan spectrum where $\zeta_0(z) = z^2 e^{-z^2}$ [10, 27]. On the other hand, it is not satisfied for the Kármán–Obukhov spectrum where $\zeta_0(z) = z^2(1+z^2)^{-7/3}$ [5, 27]. However, this is not a serious drawback since the Kármán–Obukhov spectrum is valid only in the inertial range and should be modified to decay more rapidly in the dissipative range.

Now we are ready to present the main theorem.

THEOREM 1.2. *Let* $\{x(t), y(t)\}$ *be the solutions of equations* (1.10) *with initial conditions* $\{x_0, y_0\}$ *satisfying* $\mathbb{E}(|x_0|^2+|y_0|^4) < \infty$. *Assume further that the spectrum of the Wiener process* $W(t)$ *satisfies conditions* (1.14). *Then* $\{x(t), y(t)\}$ *converge, as* $\epsilon \to 0$, *to the solutions of the limiting SDE* (1.13) $\{X(t), Y(t)\}$, *with the same initial conditions, in the following sense:*

$$\mathbb{E}\left(\sup_{0\le t\le T}\left\{|y(t)-Y(t)|^2 + |x(t)-X(t)|^2\right\}\right) \le C\,\epsilon^{2-\sigma} \tag{1.16}$$

for any $\sigma > 0$. *The constant* C *depends on the moments of the initial conditions, the spectrum of the Wiener process, the operator* A, *the maximum time* T, *and on* σ.

1.3. Discussion of the main theorem. In the main theorem we consider a model for the motion of inertial particles in a Gaussian random field model of turbulence. We study a rescaled model which describes particle motions at large times in situations where the correlation time of the fluid, and the particle/fluid time-scale ratio, are also scaled to achieve white noise behavior for the effect of the Gaussian random field on the particles. We reduce an infinite dimensional problem to an SDE in $\mathbb{T}^2\times\mathbb{R}^2$. The techniques that we describe will apply to the elimination of rapidly varying infinite dimensional OU processes in quite general situations. Loosely speaking, we study the limit $\epsilon \to 0$ for the equation

$$\frac{d^2x}{dt^2} = \frac{f(x)}{\epsilon}\eta_0\left(\frac{t}{\epsilon^2}\right) - \frac{dx}{dt},$$

with η_0 an infinite dimensional OU process with correlation time equal to $O(1)$. We obtain the limiting equations

$$\frac{d^2X}{dt^2} = \sigma(X)\frac{dW}{dt} - \frac{dX}{dt}, \tag{1.17}$$

where W is an infinite dimensional Wiener process.

It is shown in [27], using (1.3) and (1.4), that the operator $\sigma(x) = f(x)A^{-1}$ satisfies

$$\sigma(X)\sigma(X)^* = \hat{\sigma}^2\mathcal{I}, \tag{1.18}$$

and $\hat{\sigma}$ is constant, where $\mathcal{I}$ is the identity on $\mathbb{R}^2$. Hence the limiting SDE is equivalent in law to the SDE

$$\frac{d^2X}{dt^2} = \hat{\sigma}\frac{dB}{dt} - \frac{dX}{dt}. \tag{1.19}$$

The limiting equations (1.17), (1.19) arise when both the inverse fluid correlation time and the particle/fluid time-scale ratio scale linearly with the time dilation factor so that the fluid is rapidly decorrelating and the particles are heavy, giving rise to the distinguished limit studied in this paper. Note that the Itô and Stratonovich interpretations of (1.17) are the same because the multiplicative noise term appears only in the equation for particle velocity and depends only on the particle position. The particle motion described by (1.17) is equivalent in law to an OU process for the velocity $\frac{dx}{dt}$ on $\mathbb{R}^2$ (see (1.19)). This is, consequently, ergodic and gives rise to a stationary Gaussian measure for the particle velocity. This in turn yields a stationary uniform measure for the particle positions on $\mathbb{T}^2$.

Equation (1.17) has more interesting behavior when studied from the viewpoint of two-point motions. Although we have not proved it, techniques similar to those used here will prove convergence of (1.10a), (1.10b) to (1.17) as a flow.[4] Thus it is interesting to study (1.17) as a random dynamical system. We leave such a study for a future publication.

1.4. Overview. The remainder of this paper is organized as follows. In section 2 we present some background material on infinite dimensional SDEs and write the system of equations (1.8) as an abstract SDE in infinite dimensions. In section 3 we prove existence and uniqueness of solutions for the limiting SDE (1.13). In section 4 we obtain the estimates that are necessary for the proof of the convergence theorem, Theorem 1.2. In section 5 we prove the convergence theorem. Section 6 is devoted to conclusions and discussion of generalizations of the main theorem of this paper. Finally, the appendix is devoted to the derivation of various estimates for the infinite dimensional OU process (1.10c).

2. Abstract formulation. In order to carry out the subsequent analysis we shall need to use the infinite dimensional versions of the Itô lemma and the Itô isometry. It is possible to verify that (1.10) satisfies the necessary conditions for these theorems to hold. For background material on stochastic equations in infinite dimensions we refer the reader to [24].

Let H and U be two separable Hilbert spaces, and let $W(t)$ be a U-valued Q-Wiener process. Here Q is the covariance operator of $W(t)$, and the spectrum of this operator defines the spectrum of the Wiener process. Further, let $U_0 = Q^{\frac{1}{2}}(U)$, and let $L_2^0 = L_2(U_0, H)$ denote the space of Hilbert–Schmidt operators from U_0 to H. Let now $\Phi(t)$, $t \in [0,T]$, be a measurable L_2^0-valued process. Further, we define the Hilbert space $\mathcal{N}_W^2 := \mathcal{N}_W^2(0,T;L_2^0)$ consisting of all L_2^0-predictable processes $\Phi(t)$ such that $|||\Phi|||_T^2 := \mathbb{E}\int_0^t Tr[\Phi Q \Phi^*]\,ds < \infty$. Then, we can define the stochastic integral $\int_0^t \Phi(s)\,dW(s)\ \forall\,\Phi(t) \in \mathcal{N}_W^2$. This process is a continuous, square integrable H-valued martingale on $[0,T]$, and the Itô isometry holds:

$$\mathbb{E}\left|\int_0^t \Phi(s)\,dW(s)\right|^2 = |||\Phi|||_t^2 = \mathbb{E}\int_0^t Tr[\Phi Q \Phi^*]\,ds,\ \ t \in [0,T]. \tag{2.1}$$

[4] This will require further conditions on the spectrum of the Wiener process.

Let us consider now the process

$$X(t) = X(0) + \int_0^t \phi(s)\, ds + \int_0^t \Phi(s)\, dW(s), \tag{2.2}$$

where $\Phi \in \mathcal{N}_W^2$, ϕ is an H-valued predictable process Bochner integrable on $[0,T]$ **P**-a.s. and $X(0)$ is an $\mathcal{F}_0$-measurable H-valued random variable. Under these conditions the process (2.2) is well defined. Let now $F : [0,T] \times H \to \mathbb{R}$ be a uniformly continuous function on bounded subsets of $[0,T] \times H$, together with its derivatives F_t, F_x, F_{xx}.[5] Then the Itô formula holds **P**-a.s. $\forall\, t \in [0,T]$:

$$\begin{aligned} F(t, X(t)) = F(0, X(0)) &+ \int_0^t \langle F_x(s, X(s)), \Phi(s)\, dW(s)\rangle \\ &+ \int_0^t \Big\{ F_t(s, X(s)) + \langle F_x(s, X(s)), \phi(s)\rangle \\ &+ \frac{1}{2} Tr\left[F_{xx}(s, X(s)) \left(\Phi(s) Q^{\frac{1}{2}}\right) \left(\Phi(s) Q^{\frac{1}{2}}\right)^* \right] \Big\}\, ds, \end{aligned} \tag{2.3}$$

where $\langle\cdot\rangle$ denotes the inner product in H.

Now we wish to apply this abstract formulation to our problem.[6] To this end, we define the spaces $U = \mathbb{C}^K$ and $H = \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{C}^K$. H is a Hilbert space equipped with the following inner product:

$$(f_1, f_2)_H = (x_1, x_2) + (y_1, y_2) + (z_1, z_2)_{\ell^2},$$

where $f_i = [x_i\ y_i\ z_i]^T$, $i = 1, 2$ with $x_i \in \mathbb{T}^2$, $y_i \in \mathbb{R}^2$, $z_i \in \mathbb{C}^K$. Here $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\ell^2}$ denote the Euclidean and ℓ^2 inner products, respectively.

$W(t)$ is a U-valued Q-Wiener process where the covariance operator $Q : U \to U$ has eigenvalues $\{\lambda_k\}_{k\in K}$ and eigenvectors $\{\hat{e}_k\}_{k\in K}$ which form a complete orthonormal basis in $\mathbb{C}^K$. The Fourier representation of the Wiener process $W(t)$ is given by formula (1.12).

We can write (1.10) in the following abstract form:

$$d\, Z = b(Z)\, dt + V\, dW,$$

where $Z(t) = [x\ y\ \eta]^T \in H$, the drift term $b(Z) \in H$ is

$$b(Z) = \begin{bmatrix} y \\ \frac{f(x)\eta}{\epsilon} - y \\ -\frac{1}{\epsilon^2} A\, \eta \end{bmatrix},$$

and the operator $V : U \to H$ is

$$V = \begin{bmatrix} \mathbb{O}_{2\times\infty} \\ \mathbb{O}_{2\times\infty} \\ \frac{1}{\epsilon}\mathbb{I} \end{bmatrix}, \tag{2.4}$$

[5] F_x and F_{xx} are understood in the sense of Fréchet derivatives.

[6] A slightly different, though completely equivalent, formulation of system (1.10) was used in [19, sect. 4].

where $\mathbb{O}_{2\times\infty}$ is the zero operator in $L(\widehat{\mathbb{C}}^K, \mathbb{R}^2)$ and $\mathbb{I}$ is the identity operator in $L(\widehat{\mathbb{C}}^K)$. We can think of V as an infinite dimensional matrix with entries:

$$V_{ij} = \begin{cases} 0 & : \quad i = 1, \dots, 4, \ j = 1, \dots, \infty, \\ \frac{1}{\epsilon}\delta_{ij} & : \quad i = 5, \dots, \infty, \ j = 1, \dots, \infty. \end{cases}$$

In order to be able to apply the Itô lemma and Itô isometry we have to check that $V \in L_2^0 = L_2(U_0, H)$. In other words,

$$Tr[VQV^*] < \infty. \tag{2.5}$$

For the operator V defined in (2.4) condition (2.5) reduces to

$$\frac{1}{\epsilon^2} \sum_{k\in K} \lambda_k < \infty. \tag{2.6}$$

Condition (2.6) is satisfied for every fixed, finite ϵ on account of the normalization condition (1.2). The condition on the drift term is also satisfied since it is clearly Bochner integrable. Later on we will have occasion to use both formulas (2.1) and (2.3).

3. Existence and uniqueness of solutions for the limiting SDE. In this section we prove Theorem 1.1, which ensures existence and uniqueness of solutions for the limiting SDE (1.13). The proof differs from the standard existence and uniqueness proof for SDEs [18, 20] in that the system (1.13) is driven by an infinite dimensional noise.

In the following we frequently use the following lemma whose proof is straightforward and omitted for brevity.

LEMMA 3.1. *Let $D : \widehat{\mathbb{C}}^K \to \widehat{\mathbb{C}}^K$ be diagonal with entries $\{d_i\}_{i\in K} \in \mathbb{R}$. Let $G \in L(\widehat{\mathbb{C}}^K, \mathbb{R}^2)$ be defined for $\gamma \in \widehat{\mathbb{C}}^K$ by*

$$G\,\gamma = \sum_{k\in K} g_k \gamma_k, \quad g_k \in \mathbb{C}^2, \ g_k = \bar{g}_{-k}.$$

Then

$$G\,D\,(G\,D)^* = \sum_{k\in K} d_k^2 g_k g_{-k}^*.$$

Before proceeding with the proof of Theorem 1.1 we present a calculation that we need. We take the Hilbert space U to be the same as in the previous section, and we set $H = \mathbb{T}^2 \times \mathbb{R}^2$. Let $T > 0$ and $t \in [0, T]$. Then from Itô isometry (2.1), using (1.11) and Lemma 3.1 with $D = A^{-1}Q^{1/2}$ and $G = f(x)$ we have

$$\begin{aligned} \mathbb{E}\left| \int_0^t f(X(s))A^{-1}\, dW(s) \right|^2 &= \mathbb{E} \int_0^t Tr\left[\left(f(X(s))A^{-1}\right) Q \left(f(X(s))A^{-1}\right)^* \right] ds \\ &= \mathbb{E} \int_0^t \sum_{k\in K} |\nabla^\perp e_k(X(s))|^2 \frac{\lambda_k}{\alpha_k^2}\, ds \\ &\le T \sum_{k\in K} \frac{|k|^2 \lambda_k}{\alpha_k^2} \\ &\le C_1\, T, \end{aligned} \tag{3.1}$$

where $C_1 = \sum_{k\in K} \frac{|k|^2\lambda_k}{\alpha_k^2}$. This is a finite quantity under (1.14b). Similarly, in view of the Lipschitz continuity of $f(X)$, we have

$$\mathbb{E}\left|\int_0^t \left(f(X(s)) - f(\hat{X}(s))\right) A^{-1}\, dW(s)\right|^2 \le \left(\sum_{k\in K} \frac{\lambda_k\,|k|^4}{\alpha_k^2}\right) \int_0^t \mathbb{E}|X(s) - \hat{X}(s)|^2\, ds$$

$$= C_2 \int_0^t \mathbb{E}|X(s) - \hat{X}(s)|^2\, ds, \tag{3.2}$$

with $C_2 = \sum_{k\in K} \frac{|k|^4\lambda_k}{\alpha_k^2} < \infty$ on account of condition (1.14b). We also remark that, since the stochastic integral $\int_0^t f(X(s))A^{-1}\, dW(s)$ is a continuous, square integrable $\mathbb{R}^2$-valued martingale on $[0,T]$ we can use Theorem 3.8 from [24], together with (3.1), in order to obtain

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t f(X(s))A^{-1}\, dW(s)\right|^2\right) \le 4 \sup_{0\le t\le T} \mathbb{E}\left(\left|\int_0^t f(X(s))A^{-1}\, dW(s)\right|^2\right)$$

$$\le 4\, C_1\, T.$$

Similarly, we have

$$\mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^s \left(f(X(r)) - f(\hat{X}(r))\right) A^{-1}\, dW(r)\right|^2\right)$$

$$\le 4\, C_2 \int_0^t \mathbb{E}\left(\sup_{0\le r\le s}\left|X(r) - \hat{X}(r)\right|^2\right) ds. \tag{3.3}$$

Proof of Theorem 1.1. We only sketch the proof, as it is a straightforward extension of the existence and uniqueness proof for ordinary SDEs to the case of infinite dimensional noise.

We start with uniqueness. By using (3.3) we obtain

$$\mathbb{E}\left(\sup_{0\le s\le t}\left(|X(s) - \hat{X}(s)|^2 + |Y(s) - \hat{Y}(s)|^2\right)\right)$$

$$\le C \int_0^t \mathbb{E}\left(\sup_{0\le r\le s}\left(|X(r) - \hat{X}(r)|^2 + |Y(r) - \hat{Y}(r)|^2\right)\right) ds. \tag{3.4}$$

This implies uniqueness.

We now proceed with the existence part of the proof using Picard's iteration scheme. We define $X^{(0)} = X(0) = x_0$, $Y^{(0)} = Y(0) = y_0$ and define the $n+1$ term inductively as follows:

$$X^{(n+1)}(t) = x_0 + \int_0^t Y^{(n)}(s)\, ds,$$

$$Y^{(n+1)}(t) = y_0 + \int_0^t f(X^{(n)}(s))A^{-1} dW(s) - \int_0^t Y^{(n)}(s)\, ds.$$

A similar calculation to the one for the uniqueness proof yields

$$\mathbb{E}\left(|X^{(n+1)}(t)-X^{(n)}(t)|^2+|Y^{(n+1)}(t)-Y^{(n)}(t)|^2\right)\leq \frac{C^n\,t^n}{n!},\quad t\in[0,T], \tag{3.5}$$

where the constant C is a function of T, $\mathbb{E}|y_0|^2$, the spectrum of the Wiener process, and operator A. On the other hand, we have, using arguments similar to those used in (3.3) and by (3.5),

$$\mathbb{E}\left(\sup_{0\leq t\leq T}(|X^{(n+1)}(t)-X^{(n)}(t)|^2+|Y^{(n+1)}(t)-Y^{(n)}(t)|^2)\right)\leq C_3\frac{C^n\,T^n}{n!}$$

under condition (1.14c).

The rest of the existence proof follows the steps of the corresponding proof in the case of a finite dimensional Wiener process [18, Thm. 2.3.1], [20, Thm. 5.2.1]: the above estimates enable us to prove that $\{X^n(t),\,Y^n(t)\}$ converge uniformly in $[0,T]$ for a.a. ω as well as strongly in $L^2(P)$ to $\{X(t),\,Y(t)\}$. Consequently, the limit $\{X(t),\,Y(t)\}$ has all the properties mentioned in the statement of Theorem 1.1, and the only thing left to check is that they are solutions of (1.13). For this it suffices to ensure that

$$\mathbb{E}\left(\left|\int_0^t\left(f(X^n(s))-f(X(s))\right)A^{-1}\,dW(s)\right|\right)^2\to 0\ \text{ as }\ n\to\infty.$$

This follows from (3.2) and the strong convergence in $L^2(P)$ of $X^n(s)$ to $X(s)$. The proof of the theorem is now complete. □

4. Necessary estimates for the convergence theorem. In this section we obtain various estimates that we will need in the proof of the two convergence theorems. We start by integrating (1.10b) in time to obtain

$$y(t)-y(0)=\frac{1}{\epsilon}\int_0^t f(x(s))\eta(s)\,ds-\int_0^t y(s)\,ds. \tag{4.1}$$

We now use the infinite dimensional version of Itô's lemma in order to rewrite (4.1) in a more convenient form. We remark that, since the integrand in the first integral is linear in η and noise appears in (1.10) only in the equation for η, no higher order corrections will appear, and the integration by parts formula from standard calculus holds.

Before presenting the formula that results from the integration by parts let us define carefully the various operators that we will use (we shall think of η as an infinite dimensional complex vector which is bounded in the ℓ^2 norm). The operator $f(x)$ is defined by (1.11). The operator $df(x)y(s)\in L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)$ is defined by

$$\{df(x)y(s)\}\gamma=\sum_{k\in K}ik\cdot y\nabla^{\perp}e_k(x)\gamma_k. \tag{4.2}$$

The operator $A^\delta: D(A^\delta)\in\mathbb{C}^K\to\mathbb{C}^K$, $\delta>0$, is the diagonal operator with entries $\{(\alpha_k)^\delta\}_{k\in K}$. Its domain of definition consists of all elements $\eta\in\mathbb{C}^K$ for which $||A^\delta\eta||_{\ell^2}<\infty$.

The integration by parts formula gives

$$\begin{aligned} y(t) - y(0) &= \frac{1}{\epsilon}\int_0^t f(x(s))\eta(s)\,ds - \int_0^t y(s)\,ds \\ &= -\epsilon\int_0^t f(x(s))A^{-1}\,d\eta(s) + \int_0^t f(x(s))A^{-1}\,dW(s) - \int_0^t y(s)\,ds \\ &= -\epsilon\left(f(x(t))A^{-1}\eta(t) - f(x(0))A^{-1}\eta(0)\right) + \epsilon\int_0^t df(x(s))y(s)A^{-1}\eta(s)\,ds \\ &\quad + \int_0^t f(x(s))A^{-1}\,dW(s) - \int_0^t y(s)\,ds \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.3}$$

Our goal in this section is to show that the terms I_1 and I_2 are small in mean square. The terms I_3 and I_4 give the required contribution to the limiting equations.

We start by obtaining bounds on the second and fourth moments of $y(t)$. Our method will be to first obtain a bound on the fourth moment of the form,

$$\mathbb{E}|y(t)|^4 \le C\,\epsilon^{-4},$$

and then use this to obtain a uniform bound on the second moment:

$$\mathbb{E}|y(t)|^2 \le C.$$

LEMMA 4.1. *Let $y(t)$ be the solution of* (1.10)*, and assume that the initial conditions y_0 satisfy $\mathbb{E}|y_0|^4 < \infty$. Assume further that condition* (1.14e) *holds. Then we have*

$$\mathbb{E}|y(t)|^4 \le C\,\epsilon^{-4}, \tag{4.4}$$

where the constant C is a function of T, the initial conditions, the spectrum of the Wiener process, and the operator A.

Proof. Let us consider the first component of (1.10b):

$$dy_1 = \left(\frac{v_1(x(t),t)}{\epsilon} - y_1(t)\right)dt, \tag{4.5}$$

where $v_1(x,t)$ is the first component of the velocity field, $v(x,t) = f(x)\,\eta(t)$. We multiply (4.5) by $(y_1(t))^3$ to obtain

$$d(y_1(t))^4 = \left(\frac{4}{\epsilon}y_1(t)^3 v_1(x(t),t) - 4y_1(t)^4\right)dt.$$

We integrate the above equation and get

$$(y_1(t))^4 = (y_1(0))^4 + \frac{4}{\epsilon}\int_0^t (y_1(s))^3 v_1(x(s),s)\,ds - 4\int_0^t (y_1(s))^4\,ds.$$

A similar expression holds for the second component of $y(t)$. Now we have

$$\begin{aligned}|y(t)|^4 &\le 2\left((y_1(t))^4+(y_2(t))^4\right)\\ &\le 2\,|y(0)|^4+8\sum_{i=1}^{2}\left(\frac{1}{\epsilon}\int_0^t (y_i(s))^3 v_i(x(s),s)\,ds-\int_0^t (y_i(s))^4\,ds\right)\\ &\le 2\,|y(0)|^4+\frac{1}{\epsilon^4}\sum_{i=1}^{2}\int_0^t (v_i(x(s),s))^4\,ds\\ &\le 2\,|y(0)|^4+\frac{1}{\epsilon^4}\int_0^t |v(x(s),s)|^4\,ds.\end{aligned}$$

In the above derivation we used the inequality $a\,b\le\delta\,a^p+C(\delta)\,b^q$ with $p^{-1}+q^{-1}=1$ and $C(\delta)=(\delta\,p)^{-\frac{q}{p}}q^{-1}$ [4, p. 622] with $\delta=\epsilon$, $p=\frac{3}{4}$, $q=4$, $C(\epsilon)=\frac{27}{256}\epsilon^{-3}<\frac{1}{8}\epsilon^{-3}$. Now we take the expectation value of the above expression to obtain

$$\begin{aligned}\mathbb{E}|y(t)|^4 &\le 2\,\mathbb{E}|y(0)|^4+\frac{1}{\epsilon^4}\int_0^t \mathbb{E}|v(x(s),s)|^4\,ds\\ &\le C\,\epsilon^{-4},\end{aligned}$$

where the bound on the fourth moment of the velocity field which is derived in section A.2 is used. The lemma is proved. □

Now we are ready to bound the second moment of $y(t)$ uniformly in ϵ.

LEMMA 4.2. *Let $y(t)$ be the solution of* (1.10)*, and assume that the initial conditions y_0 satisfy $\mathbb{E}|y_0|^4<\infty$. Assume further that the spectrum satisfies conditions* (1.14a)*,* (1.14b)*,* (1.14c)*, and* (1.14e)*. Then we have*

$$\mathbb{E}|y(t)|^2\le C, \tag{4.6}$$

where the constant C is a function of T, the initial conditions, the spectrum of the Wiener process, and the operator A.

Proof. Step 1. The solution of (1.10b) is

$$y(t)=\frac{1}{\epsilon}e^{-t}\int_0^t e^s f(x(s))\eta(s)\,ds+y(0)e^{-t}. \tag{4.7}$$

We perform an integration by parts on the integral on the right-hand side of the above expression and then use Itô's formula for the function $G(s,x,\eta)=e^s f(x(s))A^{-1}\eta(s)$ to obtain[7]

$$\begin{aligned}y(t) \;=\;& \epsilon\,e^{-t}\int_0^t e^s f(x(s))A^{-1}\eta(s)\,ds+\epsilon e^{-t}\int_0^t e^s df(x(s))y(s)A^{-1}\eta(s)\,ds\\ &+e^{-t}\int_0^t e^s f(x(s))A^{-1}\,dW(s)-\epsilon\left(f(x(t))A^{-1}\eta(t)-e^{-t}f(x(0))A^{-1}\eta(0)\right)+y_0\,e^{-t}\\ :=\;& J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

[7]Due to the fact that $G(s,x,\eta)$ is linear in η and that only the equation for η contains a noise term, the Itô formula reduces to ordinary calculus.

Consequently, we have

$$\mathbb{E}|y(t)|^2 \leq 5\left(\mathbb{E}|J_1|^2 + \mathbb{E}|J_2|^2 + \mathbb{E}|J_3|^2 + \mathbb{E}|J_4|^2 + \mathbb{E}|J_5|^2\right). \tag{4.8}$$

We shall treat each term from the right-hand side of (4.8) separately.

Step 2. We start with the first term. A simple variant of the proof of Theorem A.2 reveals that

$$\mathbb{E}\left|f(x(s))A^{-1}\eta(s)\right|^2 = C_A < \infty \tag{4.9}$$

under condition (1.14e).[8] We now use estimate (4.9) to deduce

$$\begin{aligned}\mathbb{E}|J_1|^2 &= \epsilon^2\mathbb{E}\left|e^{-t}\int_0^t e^t f(x(s))A^{-1}\eta(s)\,ds\right|^2\\ &\leq \epsilon^2 e^{-2t}T\int_0^t e^{2s}\left|f(x(s))A^{-1}\eta(s)\right|^2 ds\\ &\leq \epsilon^2\,T^2\,C_A,\end{aligned}$$

where the constant C_A depends only on the spectrum of the Wiener process and the operator A.

Step 3. Now we proceed with the second term. We have

$$\begin{aligned}\mathbb{E}|J_2|^2 &= \epsilon^2 e^{-t}\mathbb{E}\left|\int_0^t e^s df(x(s))y(s)A^{-1}\eta(s)\,ds\right|^2\\ &\leq T\,\epsilon^2\,\mathbb{E}\int_0^T\left|df(x(s))y(s)A^{-1}\eta(s)\right|^2 ds\\ &= \epsilon^2\,\mathbb{E}\int_0^T\left|df(x(s))y(s)A^{-1}A^{-\frac{1+\delta}{2}}A^{\frac{1+\delta}{2}}\eta(s)\right|^2 ds\\ &\leq \epsilon^2\,\mathbb{E}\int_0^T ||df(x(s))y(s)A^{-\frac{3+\delta}{2}}||^2_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)}||A^{\frac{1+\delta}{2}}\eta(s)||^2_{\ell^2}\,ds\\ &= \epsilon^2\,\mathbb{E}\int_0^T \|B\|^2_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)}||A^{\frac{1+\delta}{2}}\eta(s)||_2^2\,ds,\end{aligned} \tag{4.10}$$

where $B := df(x(s))y(s)A^{-\frac{3+\delta}{2}}$ and $||\cdot||_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)}$ denotes the operator norm on $L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)$. Now we have to obtain a bound on $\|B\|_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)}$.

Step 4. The action of B on $\eta\in\widehat{\mathbb{C}}^K$ is

$$B\eta = \sum_{k\in K}[ik_2,\ -ik_1]^T e^{ik\cdot x}ik\cdot y\frac{1}{\alpha_k^{\frac{3+\delta}{2}}}\eta_k. \tag{4.11}$$

[8] In fact, the above estimate is valid under the condition $\sum_{k\in K}|k|^{4+\epsilon}\frac{\lambda_k}{\alpha_k^2} = C_A < \infty$ for some $\epsilon > 0$ which is less restrictive than (1.14e). This is because $v(x,t) = f(x(t))\,\eta(t)$, and so the bound in Theorem A.2 requires stronger decay estimates than (4.9), which contains an extra A^{-1}, and we have assumed that $\xi(z)$ grows with z.

Now we can compute the operator norm of $B : \widehat{\mathbb{C}}^K \to \mathbb{R}^2$ using Lemma 3.1:

$$\begin{aligned}\|B\|^2_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)} &= \sup_{\|\eta\|_{\ell^2}\leq 1} \|B\eta\|^2_{\mathbb{R}^2} \\ &= \sup_{\|\eta\|_{\ell^2}\leq 1} \sum_{k\in K} |k|^2 |k\cdot y|^2 \frac{1}{\alpha_k^{3+\delta}} |\eta_k|^2 \\ &\leq \sum_{k\in K} |k|^2 |k\cdot y|^2 \frac{1}{\alpha_k^{3+\delta}} \\ &\leq |y|^2 \sum_{k\in K} \frac{|k|^4}{\alpha_k^{3+\delta}}.\end{aligned}$$

The sum on the right-hand side of the above estimate is summable on account of condition (1.14c). Thus, we conclude that

$$||B||^2_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)} \leq C|y|^2, \tag{4.12}$$

where $C := \sum_{k\in K} \frac{|k|^4}{\alpha_k^{3+\delta}}$. Since the initial conditions for $\eta(t)$ are statistically stationary, the process $A^{\frac{1+\delta}{2}}\eta$ is a Gaussian process with mean zero and covariance operator $\hat{Q}\zeta = \sum_{k\in K} \alpha_k^\delta \lambda_k \zeta_k \hat{e}_k$, $\zeta = \{\zeta_k\}$. Condition (1.14a) ensures that $\hat{Q}$ is a trace class operator. Now, Corollary 2.17 from [24] enables us to bound higher order moments of Gaussian processes in terms of the second moment:

$$\begin{aligned}\mathbb{E}\|A^{\frac{1+\delta}{2}}\eta\|^4_{\ell^2} &\leq C\left(\mathrm{Tr}(\hat{Q})\right)^2 = C\left(\sum_{k\in K} \alpha_k^\delta \lambda_k\right)^2 \\ &\leq C \qquad (4.13)\end{aligned}$$

on account of condition (1.14a).

We use bounds (4.12) and (4.13) in (4.10) to obtain

$$\begin{aligned}\mathbb{E}|J_2|^2 &\leq \epsilon^2\, C\, T\, \mathbb{E}\int_0^t |y(s)|^2 ||A^{\frac{1+\delta}{2}}\eta(s)||^2_{\ell^2}\, ds \\ &\leq \epsilon^4 \frac{C\,T}{2}\int_0^t \mathbb{E}|y(s)|^4\, ds + \frac{C\,T}{2}\int_0^t \mathbb{E}||A^{\frac{1+\delta}{2}}\eta(s)||^4_{\ell^2}\, ds \\ &\leq C\,T.\end{aligned}$$

Step 5. In order to bound $\mathbb{E}|J_3|^2$ we just use Itô isometry (3.1) and Lemma 3.1:

$$\begin{aligned}\mathbb{E}|J_3|^2 &= e^{-2t}\int_0^t \mathbb{E}\left(e^s \mathrm{Tr}\left[\left(f(x(s))A^{-1}Q^{\frac{1}{2}}\right)\left(f(x(s))A^{-1}Q^{\frac{1}{2}}\right)^*\right]\right) ds \\ &\leq \left(\sum_{k\in K} \frac{\lambda_k\,|k|^2}{\alpha_k^2}\right) \leq C\,T\end{aligned}$$

on account of condition (1.14b).

Step 6. Now we consider J_4:

$$\begin{aligned}\mathbb{E}|J_4|^2 &= \epsilon^2\,\mathbb{E}\left(f(x(t))A^{-1}\eta(t) - e^{-t}f(x(0))A^{-1}\eta(0)\right)^2 \\ &\leq \epsilon^2\,\mathbb{E}\left|f(x(t))A^{-1}\eta(t)\right|^2 + 2\,\epsilon^2\,\mathbb{E}\left|f(x_0)A^{-1}\eta_0\right|^2 \\ &\leq 4\,\epsilon^2\,C_A\end{aligned}$$

on account of condition (1.14e) and (4.9).

Step 7. For J_5 we obviously have $\mathbb{E}|J_5|^2 \leq \mathbb{E}|y_0|^2$. Putting everything together we obtain

$$\mathbb{E}|y(t)|^2 \leq C,$$

and the lemma is proved. □

Now we proceed with estimating I_2 in (4.3). We have the following lemma.

LEMMA 4.3. *Assume that the spectrum of the Wiener process satisfies condition* (1.14c). *Assume further that the conditions of Lemmas* 4.1 *and* 4.2 *hold. Then for any integer* $N > 1$ *the following estimate holds:*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\epsilon\int_0^t df(x(s))y(s)A^{-1}\eta(s)\,ds\right|^2\right) \leq C\,\epsilon^{2-\sigma},$$

where $\sigma = 4/(N+1)$. *The constant* C *is a function of* T, N, *the initial conditions, the spectrum of the Wiener process, and the operator* A.

Proof. Step 1. We start with the following estimate:

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq t\leq T}|I_2(t)|^2\right) &= \epsilon^2\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t df(x(s))y(s)A^{-1}\eta(s)\,ds\right|^2\right)\\
&\leq T\,\epsilon^2\,\mathbb{E}\int_0^T\left|df(x(s))y(s)A^{-1}\eta(s)\right|^2\,ds\\
&= T\,\epsilon^2\,\mathbb{E}\int_0^T\left|df(x(s))y(s)A^{-1}A^{-\frac{1+\delta}{2}}A^{\frac{1+\delta}{2}}\eta(s)\right|^2\,ds\\
&\leq T\,\epsilon^2\,\mathbb{E}\int_0^T||df(x(s))y(s)A^{-\frac{3+\delta}{2}}||^2_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)}||A^{\frac{1+\delta}{2}}\eta(s)||^2_{\ell^2}\,ds\\
&= T\,\epsilon^2\,\mathbb{E}\int_0^T||B||^2_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)}||A^{\frac{1+\delta}{2}}\eta(s)||^2_{\ell^2}\,ds\\
&\leq C_1\,T\,\epsilon^2\int_0^T\mathbb{E}\left(|y(s)|^2||A^{\frac{1+\delta}{2}}\eta(s)||^2_{\ell^2}\right)\,ds. \qquad (4.14)
\end{aligned}$$

In the above calculations we used the bounds for the second and fourth moments of $y(t)$, estimates (4.4) and (4.6), together with the definition of the operator B and the bound (4.12).

Step 2. We fix $\gamma \in (0,1)$ and use Holder's inequality to obtain

$$\mathbb{E}\left(|y|^2||A^{\frac{1+\delta}{2}}\eta||^2_{\ell^2}\right) \leq \left(\mathbb{E}|y|^{2(1+\gamma)}\right)^{\frac{1}{1+\gamma}}\mathbb{E}\left(\|A^{\frac{1+\delta}{2}}\eta\|^{2\frac{1+\gamma}{\gamma}}_{\ell^2}\right)^{\frac{\gamma}{1+\gamma}}. \qquad (4.15)$$

Another application of Holder's inequality gives

$$\begin{aligned}
\mathbb{E}|y|^{2(1+\gamma)} &= \mathbb{E}\left(|y|^{2(1-\gamma)}|y|^{4\gamma}\right)\\
&\leq \left(\mathbb{E}|y|^2\right)^{1-\gamma}\left(\mathbb{E}|y|^4\right)^{\gamma} \leq C\epsilon^{-4\gamma}.
\end{aligned}$$

We chose $\gamma = 1/N$ with N a large integer to obtain

$$\left(\mathbb{E}|y(s)|^{2(1+\gamma)}\right)^{\frac{1}{1+\gamma}} \leq C\,\epsilon^{-\frac{4}{N+1}}. \qquad (4.16)$$

Step 3. Now we need to obtain a uniform bound on the $2(N+1)$ moment of $||A^{\frac{1+\delta}{2}}\eta(s)||_{\ell^2}$. We use [24, Cor. 2.17] as in the derivation of estimate (4.13) to obtain

$$\mathbb{E}\|A^{\frac{1+\delta}{2}}\eta\|_{\ell^2}^{2(N+1)} \leq C_N \left(\mathrm{Tr}(\hat{Q})\right)^{N+1} = C_N \left(\sum_{k\in K}\alpha_k^{\delta}\lambda_k\right)^{N+1}$$

$$\leq \hat{C}_N, \tag{4.17}$$

by condition (1.14a). We remark that this condition is independent of N.

Step 4. We use bounds (4.16) and (4.17) in (4.15) with $\gamma = 1/N$ and then use this estimate in (4.14) to obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|I_2(t)|^2\right) \leq C\,T\,\epsilon^{2-\sigma}, \tag{4.18}$$

with $\sigma = 4/(N+1)$, which can be made arbitrarily small by increasing N. This completes the proof of the lemma. □

Let us now proceed with obtaining a bound for the term I_1 in (4.3). We have the following lemma.

LEMMA 4.4. *Assume that the spectrum of the Wiener process satisfies condition* (1.14d). *Then, for ϵ sufficiently small, the following estimate holds:*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\epsilon\left(f(x(t))A^{-1}\eta(t) - f(x(0))A^{-1}\eta(0)\right)\right|^2\right) \leq C\,\epsilon^{2-\sigma}$$

for every $\sigma > 0$. The constant C depends on the spectrum of the Wiener process and the operator A.

Proof. We calculate

$$\begin{aligned} f(x)A^{-1}\eta(t) &= \sum_{k\in K}\nabla^{\perp}e_k(x)\frac{\eta_k}{\alpha_k} \\ &= \sum_{k\in K}[i\,k_2,\ -i\,k_1]^T e_k(x)\frac{\eta_k}{\alpha_k}. \end{aligned}$$

Now we use the bound (A.1), together with condition (1.14d), to obtain

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\leq t\leq T}|f(x)A^{-1}\eta(t) - f(x_0)A^{-1}\eta(t)|^2\right) \\ &\quad\leq 2\,\epsilon^2\mathbb{E}\sup_{0\leq t\leq T}|f(x)A^{-1}\eta(t)|^2 + 2\,\epsilon^2\mathbb{E}|f(x_0)A^{-1}\eta(0)|^2 \\ &\quad\leq 2\,\epsilon^2\sum_{k\in K}\frac{|k|^2}{\alpha_k^2}\mathbb{E}\sup_{0\leq t\leq T}|\eta_k(t)|^2 + 2\,\epsilon^2\sum_{k\in K}\frac{|k|^2}{\alpha_k^2}\mathbb{E}|\eta_k(0)|^2 \\ &\quad\leq 2\,\epsilon^2\sum_{k\in K}\frac{|k|^2}{\alpha_k^2}C_0\,\frac{\lambda_k}{2\alpha_k}\log\left(\frac{\alpha_k T}{\epsilon^2}\right) + 2\,\epsilon^2\sum_{k\in K}\frac{|k|^2}{\alpha_k^2}C_0\,\frac{\lambda_k}{2\alpha_k} + 2\,\epsilon^2\sum_{k\in K}\frac{|k|^2}{\alpha_k^2}\frac{\lambda_k}{2\,\alpha_k} \\ &\quad\leq C_0\,\epsilon^2\sum_{k\in K}\frac{|k|^2\lambda_k}{\alpha_k^3}\log(\alpha_k) + C_0\,\epsilon^2\sum_{k\in K}\frac{|k|^2\lambda_k}{\alpha_k^3}\log\left(\frac{T}{\epsilon^2}\right) + C\,\epsilon^2\sum_{k\in K}\frac{|k|^2\lambda_k}{\alpha_k^3} \\ &\quad\leq C_1\,\epsilon^2 + C_2\,\epsilon^2\,\log\left(\frac{T}{\epsilon^2}\right) \\ &\quad\leq C\,\epsilon^{2-\sigma} \end{aligned} \tag{4.19}$$

for every $\sigma > 0$. This completes the proof. □

5. Proof of the convergence theorem. Based on the analysis of the previous section, the integral equations for $\{x(t), y(t)\}$ can be written in the following form:

$$y(t) - y(0) = \int_0^t f(x(s))A^{-1}\, dW(s) - \int_0^t y(s)\, ds + I_2(t) + I_3(t), \tag{5.1a}$$

$$x(t) - x(0) = \int_0^t y(s)\, ds, \tag{5.1b}$$

where $\mathbb{E}(\sup_{0\le t\le T}|I_2(t)|^2) \le C\epsilon^{2-\sigma}$, $\sigma > 0$ and $\mathbb{E}(\sup_{0\le t\le T}|I_3(t)|^2) \le C\,\epsilon^2$, $\sigma > 0$. In this section we shall prove the convergence theorem, Theorem 1.2.

Proof of Theorem 1.2. The integral formulation of the limiting SDEs is

$$X(t) - x_0 = \int_0^t Y(s)\, ds,$$
$$Y(t) - y_0 = \int_0^t f(X(s))A^{-1}\, dW(s) - \int_0^t Y(s)\, ds.$$

First we bound the difference between $x(t)$ and $X(t)$:

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}|X(t) - x(t)|^2\right) &= \mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t (Y(t) - y(t))\, ds\right|\right)^2 \\ &\le T\int_0^T \mathbb{E}\left(\sup_{0\le s\le t}|Y(s) - y(s)|^2\right) dt. \end{aligned}\tag{5.2}$$

For the difference between $Y(t)$ and $y(t)$,

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le t\le T}|Y(t) - y(t)|^2\right) \\ &\quad= \mathbb{E}\left(\sup_{0\le t\le T}\left(\int_0^t (f(X(s)) - f(x(s)))A^{-1}\, dW(s)\right.\right. \\ &\qquad\left.\left. - \int_0^t (Y(s) - y(s))\, ds + I_2(t) + I_3(t)\right)^2\right) \\ &\quad\le 4\,\mathbb{E}\sup_{0\le t\le T}\left(\int_0^t (f(X(s)) - f(x(s)))A^{-1}\, dW(s)\right)^2 \\ &\qquad + 4\,\mathbb{E}\sup_{0\le t\le T}\left(\int_0^t (Y(s) - y(s))\, ds\right)^2 \\ &\qquad + 4\,\mathbb{E}\left(\sup_{0\le t\le T}|I_2(t)|^2\right) + 4\,\mathbb{E}\left(\sup_{0\le t\le T}|I_3(t)|^2\right) \\ &\quad\le C\int_0^T \mathbb{E}\left(\sup_{0\le s\le t}|X(s) - x(s)|^2\right) dt + 4\,T\int_0^T \mathbb{E}\left(\sup_{0\le s\le t}|Y(s) - y(s)|^2\right) dt \\ &\qquad + C\,\epsilon^{2-\sigma}, \end{aligned}\tag{5.3}$$

where we have assumed that the spectrum of the Wiener process satisfies condition (1.14c). We have also used the computation (3.2).

We introduce now the notation $\zeta(r) := \mathbb{E}\sup_{0\le t\le r}\left(|X(t)-x(t)|^2+|Y(t)-y(t)|^2\right)$. We combine estimates (5.2) and (5.3) to obtain

$$\zeta(T) \le C_2 \int_0^T \zeta(t)\,dt + C_1\,\epsilon^{2-\sigma}.$$

We apply Gronwall's lemma to obtain

$$\zeta(T) \le C_1\,\epsilon^{2-\sigma} e^{C_2\,T},$$

from which estimate (1.16) follows. The theorem is proved. □

6. Future work. As we have already remarked, the distinguished limit studied in this paper corresponds to the case where both inertia and friction balance the white noise term as $\epsilon \to 0$. Other distinguished limits are possible, however. Choosing $\alpha \in (1,2)$, $\beta = 3-2\alpha \in (-1,1)$, and $\gamma = \epsilon^{-\frac{2}{1+\beta}}$ in (1.7) leads to the rescaled equation:

$$\frac{d^2x}{dt^2} = \frac{f(x)}{\epsilon}\eta_0\left(\frac{t}{\epsilon^2}\right) - \epsilon^{\frac{1-\beta}{1+\beta}}\frac{dx}{dt}.$$

This distinguished limit corresponds to the case where inertia balances the white noise, but friction becomes negligible at the limit as $\epsilon \to 0$. By employing the techniques used in this paper, it is not hard to show that in this case the limiting equation is

$$\frac{d^2X}{dt^2} = \sigma(X)\frac{dW}{dt}. \tag{6.1}$$

As expected, in this case the convergence rate depends on the exponent β:

$$\mathbb{E}\sup_{0\le t\le T}\left(|x(t)-X(t)|^2+|y(t)-Y(t)|^2\right) \le C_1\epsilon^{2-\sigma} + C_2\epsilon^{2\frac{1-\beta}{1+\beta}}. \tag{6.2}$$

From (1.18) we see that (6.1) is equivalent in law to the following equation:

$$\frac{d^2X}{dt^2} = \hat{\sigma}\frac{dB}{dt}. \tag{6.3}$$

From (6.3) we deduce that the velocity will perform Brownian motion on $\mathbb{R}^2$, and hence does not have a stationary probability measure. The study of two-point motions for (6.1) is also of mathematical interest, and we will pursue this in a future publication. We note, however, that physically it is more relevant to study the problem at a longer time scale and a rescaled spatial scale and that the long time limit (6.1) is not central. The exact form of the spatial scale will be determined by the requirement that all terms in the equation of motion balance each other and appear at the asymptotic limit.

A third possible distinguished limit results from choosing $\alpha \in (-\infty, 1)$, $\beta = 1$, and $\gamma = \epsilon^{-1}$ in (1.7) which leads to the following equation:

$$\epsilon^{\delta}\frac{d^2x}{dt^2} = \frac{f(x)}{\epsilon}\eta_0\left(\frac{t}{\epsilon^2}\right) - \frac{dx}{dt}, \tag{6.4}$$

with $\delta = 1-\alpha \in (0,\infty)$. In this case the particle position converges to a diffusion process, whereas the particle velocity converges to white noise. The value of δ determines whether or not a Stratonovich or Itô interpretation should be given to the limiting

equation for the particle position. For the particular problem under investigation the limiting equation will always be interpreted as an Itô SDE, since for the inertial particles problem the Stratonovich correction disappears by (1.18) and incompressibility of the fluid velocity. Thus, the limiting SDE is

$$\frac{dX}{dt} = \sigma(X)\frac{dB}{dt} \tag{6.5}$$

$\forall\, \delta > 0$ in (6.4). Equation (6.5), on account of (1.18), implies that the inertial particles perform Brownian motion on $\mathbb{T}^2$.

The situation becomes more interesting when considering the white noise limit of Langevin equations of general type with multiplicative noise because there is a transition between the Itô and Stratonovich limits as γ passes through 2. The analysis of this problem will be presented elsewhere [23].

Finally, we mention that it is also of interest to study the model in the limit of small correlation lengths, $\ell \to 0$. In particular, studying the limits $\ell \to 0$ and $\nu \to \infty$ simultaneously is natural in the modeling of turbulent fluids.

Appendix A. Bounds on the OU process. In this appendix we shall obtain various bounds that we needed in the proof of the convergence theorem. In section A.1 we obtain a bound for $\mathbb{E}(\sup_{0\le t\le T}|\eta(t)|^2)$. In section A.2 we obtain estimates for the second and fourth moments of the velocity field.

A.1. Bound on $\mathbb{E}(\sup_{0\le t\le T}|\eta(t)|^2)$. In this section we obtain the following bound.[9]

THEOREM A.1. *Let $\eta_k(t)$ be the complex valued OU process:*

$$d\eta_k = -\frac{1}{\epsilon^2}\alpha_k\eta_k dt + \frac{1}{\epsilon}\sqrt{\lambda_k}dW_k,$$

with statistically stationary initial conditions $Re(\eta_k(0)), Im(\eta_k(0)) \in \mathcal{N}(0, \frac{\lambda_k}{2\alpha_k})$, where $W_k(t)$ is a standard complex valued Brownian motion. Then the following estimate holds:

$$\mathbb{E}\left(\sup_{0\le t\le T}|\eta_k(t)|^2\right) \le C_0\left(\frac{\lambda_k}{2\alpha_k}\log\left(\frac{\alpha_k T}{\epsilon^2}+2\right)+\frac{\lambda_k}{2\alpha_k}\right). \tag{A.1}$$

Proof. We first consider a real valued OU process with $\alpha = \lambda = 1$ and statistically stationary initial data:

$$dY = -Y\,dt + dW. \tag{A.2}$$

The process $Y(t)$ is equivalent in law to the process $X(t)$:

$$X(t) = \frac{1}{\sqrt{2}}e^{-t}W(e^{2t}).$$

To check this, note that $X(t)$ is a Gaussian process with mean zero and that, for $t > s$,

$$\mathbb{E}(X(t)X(s)) = \frac{1}{2}e^{-(t-s)}. \tag{A.3}$$

[9]The proof is due to N. O'Connell, from a private communication.

Now we have

$$\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}|Y(t)|^2 &= \mathbb{E}\sup_{0\le t\le T}|X(t)|^2 \\
&= \mathbb{E}\sup_{0\le t\le T}\left|\frac{1}{\sqrt{2}}e^{-t}W(e^{2t})\right|^2 \\
&= \mathbb{E}\sup_{1\le s\le S}\left|\frac{1}{\sqrt{2s}}W(s)\right|^2 \\
&\le \mathbb{E}\sup_{1\le s\le S}\left|\frac{W(s)}{\sqrt{2s\log\log(2+s)}}\right|^2 \log\log(2+S) \\
&\le \mathbb{E}\sup_{1\le s\le S}\left|\frac{W(s)}{\sqrt{2s\log\log(2+s)}}\right|^2 (1+\log(2+T)) \\
&:= M(S)(1+\log(2+T)), \qquad \text{(A.4)}
\end{aligned}$$

where $s := e^{2t} \Rightarrow S = e^{2T}$ and $M(S) := \mathbb{E}\sup_{1\le s\le S}|\frac{W(s)}{\sqrt{2s\log\log(2+s)}}|^2$. Moreover, we have used the inequality $\log\log(2+e^{2t}) \le 1+\log(2+t)$. Consequently, if we can prove that M is uniformly bounded independently of S, then from (A.4) we will be able to conclude that

$$\mathbb{E}\sup_{0\le t\le T}|Y(t)|^2 \le C\,\log(2+T)+C. \qquad \text{(A.5)}$$

Let us define the function

$$N(S) := \sup_{1\le s\le S}\frac{W(s)}{\sqrt{2s\log\log(2+s)}}. \qquad \text{(A.6)}$$

Now, $W(s)$ is a.s. finite in any finite time interval $[1,\, S_0]$ and so is $\frac{1}{\sqrt{2s\log\log(2+s)}}$. Thus, $N(S_0)$ is a.s. finite for any S_0. On other hand, since $N(S)$ is continuous it follows from the law of iterated logarithm that

$$N(S) \le N(\infty) := N_0 < \infty,$$

where N_0 depends on the specific realization. Since N is the supremum of a Gaussian process, from the general theory of Gaussian processes [2, Thm. 2.8] the above inequality implies

$$R(S) := \mathbb{E}N(S) \le \mathbb{E}N(\infty) = \mathbb{E}N_0 := R_0 < \infty.$$

Borell's inequality tells us that, for $\lambda > R\,(S)$, we have

$$\mathbb{P}\left(N(S) > \lambda\right) \le 2\,e^{-\frac{1}{2}\frac{(\lambda-R(S))^2}{\sigma_S^2}}, \qquad \text{(A.7)}$$

where

$$\sigma_S^2 := \sup_{1\le s\le S}\mathbb{E}\left|\frac{W(s)}{\sqrt{2s\log\log(2+s)}}\right|^2 = \frac{1}{2\log\log(3)}. \qquad \text{(A.8)}$$

We use the inequality $2\,\lambda\,R(s) \le \lambda^2\,\epsilon + \frac{1}{\epsilon}R(s)^2$ with $\epsilon = \frac{1}{2}$ in (A.7) to obtain

$$\begin{aligned}\mathbb{P}\left(N(S) > \lambda\right) &\le 2\,e^{-\frac{\lambda^2}{2\sigma_S^2}}\,e^{-\frac{R(S)^2}{2\sigma_S^2}}\,e^{\frac{2\lambda R(S)}{2\sigma_S^2}}\\ &\le 2\,e^{\frac{R(S)^2}{2\sigma_S^2}}\,e^{-\frac{\lambda^2}{4\sigma_S^2}}\\ &\le C\,e^{-\frac{\lambda^2}{4\sigma_S^2}},\end{aligned}$$

the constant C being independent of S. Now we have

$$\begin{aligned}\mathbb{P}\left(\sup_{1\le s\le S}\left|\frac{W(s)}{\sqrt{2s\log\log(2+s)}}\right|^2 > x\right) &= \mathbb{P}\left(\sup_{1\le s\le S}\left|\frac{W(s)}{\sqrt{2s\log\log(2+s)}}\right| > \sqrt{x}\right)\\ &\le 2\,\mathbb{P}\left(N(s) > \sqrt{x}\right)\\ &\le 2\,C\,e^{-\frac{x}{4\sigma_S^2}}.\end{aligned}$$

Now we can bound $M(S)$:

$$\begin{aligned}M(S) &= \int_0^\infty \mathbb{P}\left(\sup_{1\le s\le S}\left|\frac{W(s)}{\sqrt{2s\log\log(2+s)}}\right|^2 > x\right)\,dx\\ &= \int_0^{R(S)} \mathbb{P}\left(\sup_{1\le s\le S}\left|\frac{W(s)}{\sqrt{2s\log\log(2+s)}}\right|^2 > x\right)\,dx\\ &\quad + \int_{R(S)}^\infty \mathbb{P}\left(\sup_{1\le s\le S}\left|\frac{W(s)}{\sqrt{2s\log\log(2+s)}}\right|^2 > x\right)\,dx\\ &\le \int_0^{R_0} dx + 2\,C\int_0^\infty e^{-\frac{x}{4\sigma_S^2}}\,dx\\ &\le K,\end{aligned} \tag{A.9}$$

the constant K being independent of S. Thus, the bound (A.5) on the unit OU process holds.

Now we use the fact that, in law, the processes $Re(\eta_k(t))$, $Im(\eta_k(t))$, and $X(t)$ are equivalent in the following sense:

$$Re(\eta_k(t)) = \sqrt{\frac{\lambda_k}{2\alpha_k}}\,X\left(\frac{\alpha_k\,t}{\epsilon^2}\right),$$

and similarly for the imaginary part of the OU process. Consequently, an estimate of the form

$$\mathbb{E}\sup_{0\le t\le T}|X(t)|^2 \le C(T)$$

for $\eta_k(t)$ becomes

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}|\eta_k(t)|^2 &\le \mathbb{E}\sup_{0\le t\le T}|Re(\eta_k(t))|^2 + \mathbb{E}\sup_{0\le t\le T}|Im(\eta_k(t))|^2\\ &\le \frac{\lambda_k}{\alpha_k}\,C\left(\frac{\alpha_k\,T}{\epsilon^2}\right).\end{aligned}$$

Consequently, the estimate (A.5) for the rescaled OU process $\eta_k(t)$ becomes

$$\mathbb{E}\left(\sup_{0\le t\le T}|\eta_k(t)|^2\right) \le C_0\left(\frac{\lambda_k}{2\alpha_k}\log\left(\frac{\alpha_k T}{\epsilon^2}+2\right)+\frac{\lambda_k}{2\alpha_k}\right),$$

and the theorem is proved. □

Remark A.1. The above upper bound can be obtained from inequality (4.50) from [2, p. 106]. The next inequality on the same page of [2] proves that a similar lower bound holds, which proves the sharpness of estimate (A.1). Various results of this form have appeared in the literature, for example in [25, 6], however not in the explicit form that we needed for the proof of the convergence theorem.

A.2. Bounds on the moments of the velocity field. In this section we obtain bounds on the moments of the velocity field.

THEOREM A.2. *Consider the velocity field $v(x,t)=\nabla^\perp\psi(x,t)$, where the stream function $\psi(x,t)$ is the infinite dimensional OU process which is obtained from the solution of* (1.8c) *with statistically stationary initial data and $\nu_0=1$. Assume further that the spectrum of the Wiener process satisfies condition* (1.14e). *Then the second and fourth moments of the velocity field are uniformly bounded in space and time:*

$$\mathbb{E}|v(x,t)|^2 \le C_2,$$

$$\mathbb{E}|v(x,t)|^4 \le C_4.$$

Proof. First we observe that

$$\begin{aligned} v(x,t) &= \nabla^\perp\psi(x,t)\\ &= \sum_{k\in K}\nabla^\perp e^{ik\cdot x}\eta_k(t)\\ &= f(x)\eta(t), \end{aligned}$$

where the operator $f(x):\widehat{\mathbb{C}}^K\to\mathbb{R}^2$ is defined in (1.11) and $\{\eta_k\}_{k\in K}$ is an element of $\widehat{\mathbb{C}}^K$ equipped with the ℓ^2-inner product and corresponding norm. Let us first consider the second moment of the velocity field. First we compute

$$\begin{aligned} |v(x,t)|^2 &= |f(x)\eta(t)|^2\\ &= |f(x)\mathcal{B}\mathcal{B}^{-1}\eta(t)|^2\\ &\le \|f(x)\mathcal{B}\|^2_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)}\|\mathcal{B}^{-1}\eta(t)\|^2_{\ell^2}, \end{aligned}$$

where $\|\cdot\|_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)}$ denotes the operator norm and $\mathcal{B}:\widehat{\mathbb{C}}^K\to\widehat{\mathbb{C}}^K$ is the diagonal operator that multiplies by $|k|^\gamma$ the kth component of the vector on which it acts. The exponent γ is arbitrary at this point but will be determined later on. Our goal now it to obtain a bound on the operator norm of $B:=f(x)\mathcal{B}$.

The operator norm of B is defined as

$$||B||_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)} = \sup_\lambda\{\sqrt{\lambda};\ \lambda\in\sigma(B^*B)\}.$$

Now, $B^*B:\widehat{\mathbb{C}}^K\to\widehat{\mathbb{C}}^K$, which makes the computation of the spectrum difficult. However, a compact operator has the same nonzero singular values as its adjoint

[15, p. 331]. The operator B is compact, provided that $\gamma < -2$, since it can be approximated by operators of finite rank (consider a finite dimensional truncation of the sum over the lattice K). Thus, B^*B has the same nonzero eigenvalues with its adjoint $BB^* : \mathbb{C}^2 \to \mathbb{C}^2$. The problem of estimating the norm of B reduces to that of estimating the maximum eigenvalue of the 2×2 matrix $BB^* = (f(x)\mathcal{B})(f(x)\mathcal{B})^* = f(x)\mathcal{B}^2 f(x)^*$. We apply Lemma 3.1 with $G = f(x)$ and $D = \mathcal{B}$ to obtain

$$BB^* = \sum_{k\in K} |k|^{2\gamma} \begin{bmatrix} k_2^2 & -k_1k_2 \\ -k_1k_2 & k_1^2 \end{bmatrix}.$$

The trace of BB^* is

$$Tr\,(BB^*) = \sum_{k\in K} |k|^{2(\gamma+1)}. \tag{A.10}$$

Moreover, the determinant of BB^* is positive. This enables us to bound the maximum eigenvalue of BB^* by its trace:

$$\lambda_{max} \le \sum_{k\in K} |k|^{2(\gamma+1)}.$$

Consequently, we get the following bound on the operator norm of B:

$$\|B\|^2_{L(\hat{\mathbb{C}}^K,\mathbb{R}^2)} \le \sum_{k\in K} |k|^{2(\gamma+1)}.$$

Since the set K is a 2D lattice, the choice $\gamma = -2 - \frac{\epsilon}{2}$, $\epsilon > 0$, ensures that the above sum is summable:

$$\|B\|^2_{L(\hat{\mathbb{C}}^K,\mathbb{R}^2)} \le \sum_{k\in K} \frac{1}{|k|^{2+\epsilon}} = C_1 < \infty.$$

Moreover, we have

$$\|\mathcal{B}^{-1}\eta(t)\|^2_{\ell^2} = \sum_{k\in K} |k|^{4+\epsilon}|\eta_k|^2.$$

Now we can obtain a bound on the second moment of the velocity field:

$$\begin{aligned} \mathbb{E}|v(x,t)|^2 &= C_1 \sum_{k\in K} |k|^{4+\epsilon}\mathbb{E}|\eta_k|^2 \\ &= C_1 \sum_{k\in K} |k|^{4+\epsilon} \frac{\lambda_k}{2\alpha_k} \\ &:= C_2 < \infty \end{aligned}$$

on account of condition (1.14e).

Now we proceed with the bound on the fourth moment of the velocity field. Since the initial data for $\eta(t)$ are stationary, the process $\mathcal{B}^{-1}\eta(t)$ is Gaussian with mean zero and [24, Cor. 2.17] applies:

$$\begin{aligned}\mathbb{E}|v(x,t)|^4 &\le \mathbb{E}\left(\|f(x)\mathcal{B}\|^4_{L(\widehat{\mathbb{C}}^K,\mathbb{R}^2)}\|\mathcal{B}^{-1}\eta(t)\|^4_{\ell^2}\right)\\ &\le C\,\mathbb{E}\|\mathcal{B}^{-1}\eta(t)\|^4_{\ell^2}\\ &= \hat{C}\,\left(\mathbb{E}\|\mathcal{B}^{-1}\eta(t)\|^2_{\ell^2}\right)^2\\ &\le C_4.\end{aligned}$$

The proof of the theorem is now complete. □

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