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# INERTIAL PARTICLES IN A RANDOM FIELD

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The motion of an inertial particle in a Gaussian random field is studied. This is a model for the phenomenon of preferential concentration, whereby inertial particles in a turbulent flow can correlate significantly. Mathematically the motion is described by Newton's second law for a particle on a 2-D torus, with force proportional to the difference between a background fluid velocity and the particle velocity itself. The fluid velocity is defined through a linear stochastic PDE of Ornstein–Uhlenbeck type. The properties of the model are studied in terms of the covariance of the noise which drives the stochastic PDE.

Sufficient conditions are found for almost sure existence and uniqueness of particle paths, and for a random dynamical system with a global random attractor. The random attractor is illustrated by means of a numerical experiment, and the relevance of the random attractor for the understanding of particle distributions is highlighted.

*Keywords*: Inertial particles; Ornstein–Uhlenbeck process; random velocity field; preferential concentration.

AMS Subject Classification: 37H10, 37M25, 60H15, 60H35

## 1. Introduction

There is considerable evidence, both experimental and numerical, showing that the distribution of inertial particles in a turbulent velocity field is highly correlated with the turbulent motions, [12–14, 17, 30, 32, 33–35].

Figure 1, taken from paper [17], shows the distribution of lycopodium particles in a turbulent fluid. Figure 2, also from [17], is obtained by overlaying a square grid on Fig. 1 and counting the number of particles inside each square of the grid. The resulting histogram quantifies the observation that, in Fig. 1, there is substantial area where particle density is very low, and where it is very high, when

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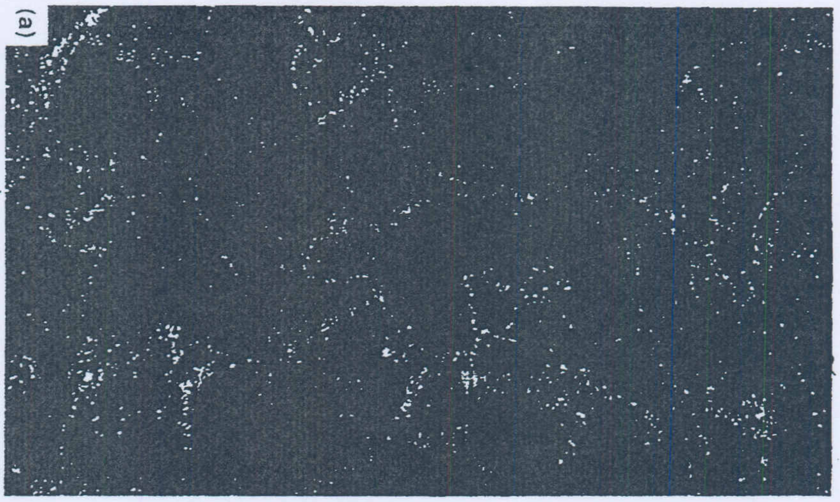


Fig. 1. Photograph of 28 $\mu$m Lycopodium particles illuminated by a laser sheet on the centerplane of a vertical turbulent channel flow [17]. (Reprinted with permission from [17].)

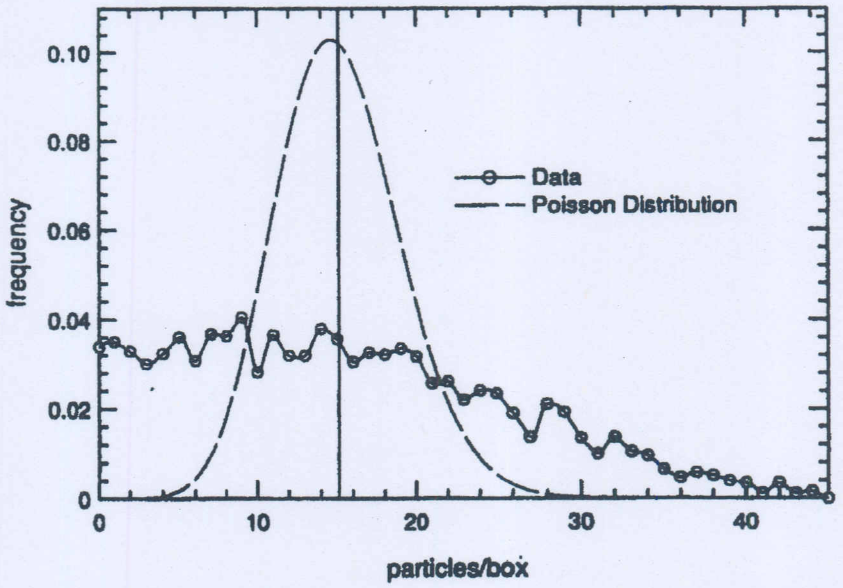


Fig. 2. Distribution of particle number density for 28 $\mu$m Lycopodium particles, St = 0.7 on a 2 mm square grid. Also plotted is Poisson, or random distribution, for the same mean number of particles per box [17]. (Reprinted with permission from [17].)

compared with the Poisson distribution; this phenomenon is known as *preferential concentration.*

In paper [32] we propose a model to investigate preferential concentration. The model exhibits good agreement with experimental data, allows efficient numerical simulation of large number of particles, including the effect of inter-particle collisions if desired, and is, to a certain extent, amenable to mathematical analysis. The model consists of Stokes's law for the particle motion:

$$\tau\ddot{x}(t) = v(x,t) - \dot{x}(t)$$

with the background velocity field $v$ being modelled by a linear stochastic PDE for a stream function, generating an incompressible velocity field.

The aim of this paper is to highlight the interesting properties of such particle motion laws, and to show that the language of random dynamical systems provides a natural framework for their study. The mathematics is straightforward building, to a large extent, on material in [8]. However the numerical experiments show very interesting mathematical structure and suggest that more detailed mathematical study is called for. In Sec. 2 we describe the model itself. Section 3 summarizes the existence, uniqueness and regularity properties of the stream function, utilizing the theory in [8, 9]. In Sec. 4 we study particle motions as a random dynamical system, proving the existence of a global random attractor. In Sec. 5 we conclude with a numerical experiment illustrating the properties of the random attractor.

The papers Kraichnan [24], Careta *et al.* [2], Martí *et al.* [28], Juneja *et al.* [20], the review article Majda and Kramer [26] and the book [18, pp. 108–113], contain useful discussions of the creation of random fields which mimic turbulence. Our velocity field $v$ is most closely related to the one constructed in [18]. There is very little mathematical analysis of inertial particles in a random field. However, there is considerable literature concerned with the study of fluid particles, or passive tracers, in the non-inertial context $\tau = 0$. Relevant references include Taylor [36], Kraichnan [24], Fannjiang and Komorowski [15], Komorowski and Papanicolaou [22] and Carmona and Xu [4]. The review article [26] gives extensive background on the subject.

In this paper we are interested in $N$-point motions in a random field; in particular we are interested in the case of $N \gg 1$ so that we can study particle distributions in a meaningful way. Thus we are studying a stochastic flow [25], and the study of two-point motions plays a central role. There is some literature on the topic of two-point motion for non-inertial passive tracers, see [11, 16, 21]. The papers by Carmona *et al.* [3], Cranston *at al.* [5] and Dolgopyat *et al.* [10] consider the evolution of sets of initial conditions under the stochastic flow induced by passive tracer motion. Most studies of passive tracers in a random velocity field concern incompressible flow fields. Recently Gawedski and Vergassola [19] have initiated study of particle tracer correlations in compressible random flows; this has a relation to the model we study which, when viewed in the phase space of particle positions and velocities, is driven by a vector field with nonzero divergence.

## 2. Mathematical Model

In this section we describe the model. Let $K = 2\pi\mathbb{Z}^2\backslash\{(0,0)\}$, $e_k(x) = e^{ik\cdot x}$ and let $\{\beta_k\}_{k\in K}$ be a sequence of standard *complex-valued* Brownian motions; this means

$\operatorname{Re}\beta_k$ and $\operatorname{Im}\beta_k$ are independent real-valued Brownian motions, each with variance $1/2$ and different $\beta_k$ are independent except that $\beta_{-k} = \beta_k^*$. For $k \in K$ choose $\lambda_k = \zeta(|k|)$ for some $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$\int_{\mathbb{R}} z\zeta(z)dz = 1\,. \tag{2.1}$$

In non-dimensional variables our model for a particle with position $x$ in a velocity field $v$ is

$$\begin{aligned} \tau\ddot{x}(t) &= v(x(t),t) - \dot{x}(t)\,, \\ v &= \nabla^\perp\psi, \\ \frac{\partial\psi}{\partial t} &= \nu\Delta\psi + \sqrt{\nu}\frac{\partial W}{\partial t}\,, \\ W(t,x) &= \sum_{k\in K}\sqrt{\lambda_k}e_k(x)\beta_k(t)\,. \end{aligned} \tag{2.2}$$

Equations (2.2) are augmented with periodic boundary conditions in $x \in \mathbb{T}^2$, and initial data $(x_0, v_0, \psi_0)$ for $(x, \dot{x}, \psi)$. We usually choose initial data so that $\psi$ is stationary. This ensures that $v$ itself is stationary.

The PDE for $\psi$ is an infinite dimensional Ornstein–Uhlenbeck (OU) process. To get an idea of the solution we use separation of variables, writing

$$\psi(t,x) = \sum_{k\in K}\hat{\psi}_k(t)e_k(x)\,, \tag{2.3}$$

for $\hat{\psi}_k : \mathbb{R} \to \mathbb{C}$, $k \in K$. Then

$$v(x,t) = \sum_{k\in K}\hat{\psi}_k(t)\nabla^\perp e_k(x) := \sum_{k\in K}\hat{v}_k(t)e_k(x)\,.$$

Let $\alpha_k = |k|^2$ so that $\Delta e_k = -\alpha_k e_k$. It follows that

$$\hat{\psi}_k(t) = e^{-\nu\alpha_k t}\langle\psi_0, e_k\rangle + \sqrt{\nu\lambda_k}\int_0^t e^{-\nu\alpha_k(t-s)}d\beta_k(s)\,. \tag{2.4}$$

If we separate the series into the deterministic part $\psi^0(t,x)$ due to the initial conditions, and a stochastic part $\psi^1(t,x)$, then the Fourier coefficients of the two functions are

$$\hat{\psi}_k^0(t) := e^{-\nu\alpha_k t}\langle\psi_0, e_k\rangle\,, \qquad \hat{\psi}_k^1(t) := \sqrt{\lambda_k\nu}\int_0^t e^{-\nu\alpha_k(t-s)}d\beta_k(s)\,. \tag{2.5}$$

We will use the fact that the variance of $\hat{\psi}_k^1$ is

$$\mathbb{E}|\hat{\psi}_k^1(t)|^2 = \frac{\lambda_k}{2\alpha_k}(1 - e^{-2\alpha_k t}) \tag{2.6}$$

which is a simple application of the Itô isometry.

If we choose initial data so that each Fourier component of $\psi$ is stationary, then

$$\hat{\psi}_k(t) = \sqrt{\nu\lambda_k}\int_{-\infty}^t e^{-\nu\alpha_k(t-s)}d\beta_k(s)\,. \tag{2.7}$$

For this stationary solution

$$\mathbb{E}|\hat{\psi}_k(t)|^2 = \frac{\lambda_k}{2\alpha_k} \Rightarrow \mathbb{E}|\hat{v}_k|^2 = \frac{1}{2}\lambda_k\,. \tag{2.8}$$

Choosing $\lambda_k$ defines the energy spectrum. Note also that

$$\mathbb{E}|v(x,t)|^2 = \frac{1}{2}\sum_{k\in K}\lambda_k = \frac{1}{2}\sum_{k\in K}\zeta(|k|) \approx \frac{1}{4\pi}\int_{\mathbb{R}} z\zeta(z)dz = \frac{1}{4\pi}\,.$$

Choosing $\zeta(z) = \ell^2\zeta_0(\ell z)$ with

$$\int_{\mathbb{R}} z\zeta_0(z)dz = 1$$

ensures the normalization (2.1); the function $\zeta_0$ gives the shape of the energy spectrum, and $\ell$ sets the length-scale. Two choices that we have used in our numerical studies, here and in [32], are the *Kraichnan spectrum* [24]

$$\zeta_0(z) = z^2 e^{-|z|^2}\,,$$

and the *Kármán–Obukhov spectrum* [18]

$$\zeta_0(z) = z^2(1+z^2)^{-7/3}\,.$$

Given a shape $\zeta_0$ for the spectrum, typically with a single peak at $k_0$, three parameters remain in the problem:

$\tau$ particle time constant/fluid time constant;
$\frac{1}{\nu}$ non-dimensional velocity correlation time;
$\ell$ non-dimensional correlation length – where $\zeta$ peaks at $\mathcal{O}(1/\ell)$.

In this paper we consider $\tau > 0$. The phenomenon of preferential concentration is observed experimentally, and numerically, only for $\tau$ of order one [12, 32].

The limit of rapidly decorrelating velocity field, $\nu \to \infty$, is of some interest and we study this in [32]. In the qualitative analyzes we present here in Secs. 3 and 4, $\tau$ plays no significant role, provided it is positive; however the singular limit $\tau \to 0$ is of some interest and we will study this limit in future work. The study of $N$-point motions as $\ell \to 0$, requiring spatial homogenization, is also of interest, and we also leave this for future study.

## 3. The Linear Stochastic PDE

In this section we study existence, uniqueness, stationarity and regularity of solutions to the linear SPDE arising in (2.2). We set up this stochastic PDE as an ODE in a Hilbert space, to facilitate statement of the existence, uniqueness and regularity results for $\psi$. Most of these results follow from material in [8, 9] and are collected together here for convenience. As $\nu$ plays no significant role here and in Sec. 4 we set it to 1. Abstractly we write (2.2) as

$$d\psi(t) = A\psi(t)dt + dW(t)\,, \quad \psi(0) = \psi_0\,, \tag{3.1}$$

$$\tau\ddot{x}(t) = v(x(t),t) - \dot{x}(t)\,, \quad x(0) = x_0,\ \dot{x}(0) = v_0\,, \tag{3.2}$$

where $v = \nabla^\perp\psi$, and $A = \Delta$ is the Laplacian in the unit square $\mathcal{O} = [0,1] \times [0,1]$ in $\mathbb{R}^2$ with periodic boundary conditions. The operator $A$ sets the relative decay of correlation in different length scales of $v$, which we have fixed by choosing $A = \Delta$. However, since other choices may be of interest, we formulate our results here in terms of an arbitrary negative-definite, self-adjoint $A$.

Set $e_k(x) = e^{ik\cdot x}$ for $k \in K = 2\pi\mathbb{Z}^2\backslash\{(0,0)\}$. Then $\{e_k\}_{k\in K}$ are eigenfunctions of $A$, and

$$Ae_k = -\alpha_k e_k\,.$$

For $A = \Delta$ we have $\alpha_k = |k|^2 := k_1^2 + k_2^2$. The set $\{e_k\}_{k\in K}$ is an orthonormal basis for the Hilbert space

$$H = \left\{\psi \in L^2_{\text{per}}(\mathcal{O}) : \int_{\mathcal{O}} \psi(x)dx = 0\right\}$$

equipped with the usual inner product and norm

$$\langle u, v\rangle = \int_{\mathcal{O}} u(x)v^*(x)dx\,, \quad u, v \in H\,, \quad \|u\|^2 = \int_{\mathcal{O}} |u(x)|^2 dx\,,$$

where $v^*$ denotes the complex conjugate of $v$. We denote the Sobolev spaces of periodic functions whose $s^{\text{th}}$ derivative is in $H$ by $H^s$. The domain of $A$, $D(A)$, is $H^2$.

$W$ is a Wiener process in $H$ meaning that it is an $H$-valued stochastic process, defined by the expansion

$$W(t) = \sum_{k\in K} \sqrt{\lambda_k}\beta_k(t)e_k\,.$$

Since we have chosen $\lambda_{-k} = \lambda_k$ and $\beta_{-k} = \beta_k^*$ we see that $W$ is real-valued.

We will find it convenient to define the operator $Q : H \to H$ as the operator having eigenvectors $\{e_k\}_{k\in K}$ and spectrum $\{\lambda_k\}_{k\in K}$,

$$Qe_k = \lambda_k e_k\,.$$

We refer to $Q$ as the *covariance operator* of $W$, since for any $u, v \in H$,

$$\mathbb{E}\langle W(t), u\rangle\langle W(t), v\rangle^* = t\langle Qu, v\rangle\,.$$

(See Proposition 4.1 in [8] where this result is established.)

The spectrum of $Q$ plays a central role in the analysis. It can be chosen so that the resulting energy spectrum of $v = \nabla^\perp\psi$ matches conjectured forms for 2D turbulent flows. Furthermore, its decay as $|k| \to +\infty$ determines the regularity of the velocity field $v$. In this context it is convenient to define the operators $Q_t : H \to H$ and $Q_\infty : H \to H$ by

$$Q_t e_k = (1 - e^{-2\alpha_k t})\frac{\lambda_k}{2\alpha_k} e_k \qquad \text{and} \qquad Q_\infty e_k = \frac{\lambda_k}{2\alpha_k} e_k \,. \tag{3.3}$$

The conditions that $Q_t$ or $Q_\infty$ are trace class, $\operatorname{Tr} Q_t < +\infty$ or $\operatorname{Tr} Q_\infty < +\infty$, are both equivalent to

$$\sum_{k\in K} \frac{\lambda_k}{2\alpha_k} < +\infty \,.$$

By a *weak solution* of (3.1) we mean a process $\psi : [0,T] \to H$ such that the trajectories of $\psi$ are a.s. Bochner integrable and

$$\langle \psi(t), \zeta \rangle = \langle \psi_0, \zeta \rangle + \int_0^t \langle \psi(s), A^* \zeta \rangle ds + \langle W(t), \zeta \rangle \qquad \forall \zeta \in D(A^*) \,.$$

Formally the solution is given by (2.3) with Fourier coefficients given by (2.4) with $\nu = 1$. Note that from (2.5)–(2.8) it follows that $\hat{\psi}_k$ is an ergodic SDE with invariant measure a Gaussian $\mathcal{N}(0, \lambda_k / 2\alpha_k)$. This strongly suggests that the SPDE has a unique invariant measure on $H$, namely the mean zero Gaussian with covariance operator $Q_\infty$.

**Theorem 3.1.** *Assume* $\operatorname{Tr} Q_\infty = \sum_{k\in K} \frac{\lambda_k}{2\alpha_k} < +\infty$. *If* $\psi_0 \in H$ *then Eq.* (3.1) *has a unique weak solution. Furthermore, Eq.* (3.1) *has a unique invariant measure* $\nu$ *on* $H$ *which is Gaussian with mean zero and covariance operator* $Q_\infty$. *The family of measures* $\{\mu_t\}_{t\geq 0}$ *on* $H$ *induced by the weak solution of* (3.1) *is tight and* $\mu_t \Rightarrow \nu$ *in* $H$ *as* $t \to \infty$.

**Proof.** Theorem 5.4 in [8] gives existence and uniqueness of weak solutions. Existence of the invariant measure follows from Theorem 6.2.1 of [9] and convergence to it is a straightforward consequence; see [31]. □

In the following theorem we slightly abuse notation and write $X \in \mathcal{C}$ for a Gaussian process $X$ to mean that *there is a Gaussian process with the same covariance as* $X$ *that has a version with sample paths in* $\mathcal{C}$.

**Theorem 3.2.** *Let* $\psi$ *be the stationary Ornstein–Uhlenbeck process solving* (3.1) *and assume that*

$$\sum_{k\in K} \frac{\lambda_k}{2\alpha_k} |k|^{2m+2\gamma} < +\infty$$

*for some* $\gamma \in (0,1)$. *Then* $\psi \in C^{\gamma/2-\varepsilon}(\mathbb{R}; C^{m+\gamma/2-\varepsilon}(\mathcal{O}))$, *any* $\varepsilon > 0$.

**Proof.** This follows by a straightforward generalization of Theorem 5.20 in [8]; see [31] for details. □

The following corollary is our main use of this theorem. We consider particle paths given by (3.2) and ask, for a given realization of $v$, whether there is a unique local solution $(x, \dot{x}) \in C^1([0,T], \mathbb{T}^2 \times \mathbb{R}^2)$ for some time $T$ which depends upon realization.

**Corollary 3.1.** *If $\sum_{k\in K} \frac{\lambda_k}{2\alpha_k}|k|^{4+\varepsilon} < +\infty$ for some $\varepsilon > 0$ then* (3.2) *has a unique local solution a.s.*

**Proof.** Set $\varepsilon = \gamma/2$ and $m = 2$ in Theorem 3.2 to get $\psi \in C(\mathbb{R}; C^2(\mathcal{O}))$ which suffices for (3.2) to have a unique solution, since then $v = \nabla^\perp\psi$ is Lipschitz in space and continuous in time. □

Applying Theorem 3.2 to an algebraically decaying spectrum for $Q$ gives the following corollary.

**Corollary 3.2.** *Let $\psi$ be the stationary Ornstein–Uhlenbeck process solving* (3.1) *and assume that $\lambda_k \sim |k|^{-s}$ for some $s > 0$ and that $\alpha_k \sim |k|^2$. Then*

$$\psi \in C^{\frac{\gamma}{2}-\varepsilon}(\mathbb{R}; C^{m+\frac{\gamma}{2}-\varepsilon}(\mathcal{O}))$$

*for $m = \lfloor \frac{s}{2} \rfloor$ and $\gamma = \frac{s}{2} - m$.*

For the *Kraichnan spectrum* we have $\lambda_k \sim |k|^2 e^{-|k|^2}$ and it follows that $\psi \in C^{\frac{1}{2}-\varepsilon}(\mathbb{R}^+; C^m(\mathcal{O}))$ for any $m$. This implies almost surely the existence and uniqueness for the ODE (3.2) governing particle motions.

For the *Kármán–Obukhov spectrum* $\lambda_k \sim |k|^2(1+|k|^2)^{-7/3}$. Here $\lambda_k \sim |k|^{-8/3}$ so applying Corollary 3.2 gives $\psi \in C^{\frac{1}{6}-\varepsilon}(\mathbb{R}^+; C^{1+\frac{1}{6}-\varepsilon}(\mathcal{O}))$. Thus we are not able to prove that the Kármán–Obukhov spectrum is regular enough for (3.2) to have a unique solution.

By treating time and space differently it is likely that one can get the stronger result that $\psi \in C^{\frac{\gamma}{2}-\varepsilon}(\mathbb{R}; C^{m+\gamma-\varepsilon}(\mathcal{O}))$ under the conditions of Theorem 3.2. However, even this does not give enough regularity for a unique solution of (3.2) in the Kármán–Obukhov case. Generalized notions of solution for particle trajectories have recently been introduced for the passive tracer problem in situations where the velocity field is not Lipschitz; see, for example, [23]. This idea could be adapted to the model for inertial particles studied here.

By a *strong solution* to (3.1) we mean a process $\psi : [0,T] \to H$ that satisfies (3.1) in the sense that

$$\psi(t) = \psi_0 + \int_0^t A\psi(s)ds + W(t)$$

for all $t \in (0,T)$. For this to make sense we require in addition that, almost surely, $\psi(t) \in D(A)$, the domain of $A$, for all $t \in [0,T]$, and $A\psi \in L^1(\mathbb{R}^+; H)$.

**Theorem 3.3.** *Assume $\sum_{k\in K} \lambda_k |k|^2 < \infty$ and consider the case where $A = \Delta$ on $\mathbb{T}^2$. If $\psi_0 \in D(A)$, then Eq.* (3.1) *has a unique strong solution.*

**Proof.** See Theorem 5.29 in [8]. □

When studying random attractors in the next section, the following generalization of Theorem 3.3 will be of interest. Notice that, when $A = \Delta, \Theta = (-A)^s v$ solves the linear SPDE

$$\begin{aligned} \frac{\partial \Theta}{\partial t} &= A\Theta + \frac{\partial N}{\partial t}, \\ N(t,x) &= \sum_{k \in K} i\sqrt{\lambda_k}(k_2, -k_1)^{\mathrm{T}} |k|^{2s} e_k(x) \beta_k(t). \end{aligned} \tag{3.4}$$

For future reference we observe that the two components of the noise $N$ are trace class Wiener processes if

$$\sum_{k \in K} \lambda_k |k|^{2(1+2s)} < \infty. \tag{3.5}$$

Application of Theorem 5.29 in [8] gives the following result.

**Theorem 3.4.** *Assume $\sum_{k \in K} \lambda_k |k|^{4(1+s)} < \infty$ and consider the case where $A = \Delta$ on $\mathbb{T}^2$. Let $v_0 = \nabla^{\perp}\psi_0$ and assume that $(-A)^s v_0 \in D(A)$. Then Eq. (3.4) has a unique strong solution $\Theta(t) = (-A)^s v(t) \in D(A)$.*

## 4. Particle Motion as a Random Dynamical System

In this section we exhibit spectral conditions on $Q$ under which (3.2) has a global random attractor. We use the formalism of random dynamical systems (RDS) in the sense of Arnold [1]. The key technical issue is to derive logarithmic bounds on paths of $\psi$. In [29] it is shown that if (3.1) is finite dimensional (initial data compactly supported in Fourier space) then the random dynamical systems defined by (3.2) is ergodic. Work is currently in progress to extend this result to the infinite dimensional case under consideration here. The invariant measure for $(x, \dot{x})$ is supported on the random attractor [6].

Throughout this section we assume that

$$\sum_{k \in K} \lambda_k |k|^{6+\varepsilon} < \infty \tag{4.1}$$

for some $\varepsilon > 0$. By Theorem 3.2 this implies that all stationary solutions of (3.1) satisfy

$$\psi \in \Omega := C(\mathbb{R}, C^{4+\delta}(\mathcal{O}))$$

for any $\delta \in (0, \varepsilon/4)$. Throughout the following we fix $\delta$ in this open interval.

### 4.1. *Setup as a RDS*

Recall that a measurable mapping $\varphi : \mathcal{T} \times \Omega \times X \to X$ is a *random dynamical system* if it is a cocycle over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ called the *noise* [1].

To model the stationary noise $\psi$ from (3.1) which drives the solution to Eq. (3.2), we take the following metric dynamical system, denoted above by $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$:

1. For $\Omega$ we take $\Omega = C(\mathbb{R}; C^{4+\delta}(\mathcal{O}))$; then (3.2) has a unique (local) solution for every $\psi \in \Omega$.
2. For the $\sigma$-algebra $\mathcal{F}$ we take the Borel $\sigma$-algebra on $\Omega$.
3. For $\mathbb{P}$ we take the probability measure induced on $\Omega$ by the unique invariant measure $\mu$ on $H$ for the SPDE (3.1); under (4.1) this is supported on $\Omega$.
4. We let the dynamical system $\theta$ be the shift map,

$$\theta_s\psi(t) = \psi(t+s) \quad \text{for } \psi \in \Omega\,.$$

For the RDS we set the time $\mathcal{T} = \mathbb{R}$, the state space $X = \mathbb{T}^2 \times \mathbb{R}^2$ where $\mathbb{T}$ is the unit circle (so $\mathbb{T}^2$ is the 2D torus), and define the cocycle $\varphi$ as the solution to (3.2). That is, for $\varphi(t,\psi)x = \varphi(t,\psi,x)$,

$$\varphi(t,\psi)(x_0,v_0) = (x(t),\dot{x}(t))\,,$$

where $(x(t),\dot{x}(t)) \in \mathbb{T}^2 \times \mathbb{R}^2$ is the solution at time $t$ to

$$\tau\ddot{x}(t) = \nabla^{\perp}\psi(x(t),t) - \dot{x}(t)\,, \quad x(0) = x_0\,, \quad \dot{x}(0) = v_0\,. \tag{4.2}$$

Implicit here is the assumption that (3.2) has a unique solution, locally in time, for which it suffices that $\psi \in C(\mathbb{R}; C^2(\mathcal{O}))$; this holds for all $\psi \in \Omega$. In fact the solution is global on $\mathbb{R}$ as the techniques giving a global random attractor show. Thus the co-cycle property

$$\varphi(t+s,\psi) = \varphi(t,\theta_s\psi) \circ \varphi(s,\psi) \quad \forall\, s,t \in \mathbb{R}, \psi \in \Omega$$

holds identically.

### 4.2. *Random attractor*

We review the standard framework for attractors of a RDS. A random set $A$ *attracts* another random set $B$ if

$$\lim_{t\to+\infty} d(\varphi(t,\theta_{-t}\psi)B(\theta_{-t}\psi), A(\psi)) = 0 \quad \text{a.s.}$$

Here $d$ is the nonsymmetric distance

$$d(B,A) := \sup_{x\in B}\inf_{y\in A} d(x,y)\,,$$

where $d$ is the metric on $X$. A random set $K$ *absorbs* the random set $B$ if for almost all $\psi$ there exists a time $t_B(\psi)$ such that

$$\varphi(t,\theta_{-t}\psi)B(\theta_{-t}\psi) \subset K(\psi) \quad \forall\, t \geq t_B(\psi)\,.$$

We say that a random set $A$ is *strictly $\varphi$-forward invariant* if

$$\varphi(t,\psi)A(\psi) = A(\theta_t\psi) \quad \forall\, t > 0\,.$$

As this is the only notion of invariance that we will deal with, we refer to such a set as simply *invariant.* A compact random set is a *global random attractor* for $\varphi$ if it is invariant and attracts every bounded (non-random) subset of $X$. Global random attractors are unique when they exist [6]. It is proved in [7] that, if there exists a compact random set absorbing all bounded non-random sets, then there exists a global random attractor. That is, to prove existence of an attractor it is enough to show that for any $\psi \in \Omega$ there exists a compact set $K = K(\psi)$ such that for any bounded non-random set $B \subset X$ there is a $t^*(B, \psi) > 0$ such that

$$\varphi(t, \theta_{-t}\psi)B \subset K(\psi) \quad \forall\, t \geq t^*(B, \psi)\,.$$

We use this approach to establish the existence of a random attractor for (3.2). Recall that the state space is $X = \mathbb{T}^2 \times \mathbb{R}^2$. For any $B \subset X$, $\varphi(t, \theta_{-t}\psi)B$ is the set

$$\left\{ \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} : \tau\ddot{x}(s) = v(x(s), s-t) - \dot{x}(s),\ \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \in B \right\}.$$

$X$ is compact in the position coordinate so it is only necessary to establish that $\dot{x}(t)$ is ultimately bounded independently of data. Taking the inner product of the equation

$$\tau\ddot{x}(s) = v(x(s), s-t) - \dot{x}(s)$$

with $\dot{x}(s)$ and using $2\,x \cdot y \leq |x|^2 + |y|^2$, where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$, gives

$$\begin{aligned}
\tau\frac{1}{2}\frac{d}{ds}|\dot{x}(s)|^2 &= \dot{x}(s) \cdot v(x(s), s-t) - |\dot{x}(s)|^2 \\
&\leq \frac{1}{2}|v(x(s), s-t)|^2 + \frac{1}{2}|\dot{x}(s)|^2 - |\dot{x}(s)|^2 \\
&= \frac{1}{2}|v(x(s), s-t)|^2 - \frac{1}{2}|\dot{x}(s)|^2\,.
\end{aligned}$$

Multiplying by $\frac{2}{\tau}e^{s/\tau}$ on each side gives

$$\frac{d}{ds}e^{s/\tau}|\dot{x}(s)|^2 \leq \frac{1}{\tau}e^{s/\tau}|v(x(s), s-t)|^2\,. \tag{4.3}$$

Now assume for the moment that there exist, almost surely, $C(\psi), D(\psi) > 0$ such that

$$\sup_{x \in \mathcal{O}} |v(x,t)|^2 \leq C + D\log(1 + |t|) \qquad \forall\, t \leq 0\,. \tag{4.4}$$

In the next subsection (Corollary 4.1) we will show that, provided that $A$ is the Laplacian on $\mathbb{T}^2$ and (4.1) holds, this is indeed the case. Note that $C$ and $D$ depend on the realization of the OU process, but not on time $t$. Integrating the inequality (4.3) from 0 to $t$ we get

$$\begin{aligned}
e^{t/\tau}|\dot{x}(t)|^2 - |\dot{x}(0)|^2 &\leq \frac{1}{\tau}\int_0^t e^{s/\tau}|v(x(s), s-t)|^2 ds \\
&\leq \frac{1}{\tau}\int_0^t e^{s/\tau}[C + D\log(1 + |s-t|)]ds
\end{aligned}$$

so that

$$\begin{aligned}|\dot{x}(t)|^2 &\le e^{-t/\tau}|\dot{x}(0)|^2 + \frac{1}{\tau}\int_0^t e^{-(t-s)/\tau}[C + D\log(1+(t-s))]ds \\ &= e^{-t/\tau}|\dot{x}(0)|^2 + \frac{1}{\tau}\int_0^t e^{-u/\tau}[C + D\log(1+u)]du \\ &\le e^{-t/\tau}|\dot{x}(0)|^2 + \frac{1}{\tau}\int_0^{+\infty} e^{-u/\tau}[C + D\log(1+u)]du \\ &\le e^{-t/\tau}|\dot{x}(0)|^2 + C + D\log(1+\tau)\,.\end{aligned}$$

The last inequality is an application of Jensen's inequality. This gives the required absorption in $X$ and hence the following:

**Theorem 4.1.** *If $\sum_{k\in K}\lambda_k|k|^{6+\varepsilon} < +\infty$ for some $\varepsilon > 0$ then the random dynamical system defined by* (3.2), *with $A$ the Laplacian on $\mathbb{T}^2$, has a global random attractor.*

### 4.3. *Almost sure pathwise estimates for $\psi$*

To complete the proof of existence of a global random attractor it remains to prove that, almost surely, there are constants $C$ and $D$ such that $v$ satisfies (4.4). We use the following result concerning OU processes.

**Theorem 4.2.** *Assume that there is a unique strong solution to*

$$dX = AX dt + dW\,, \tag{4.5}$$

*where $A : H \to H$ is a linear operator on a separable Hilbert space $H$, and $W$ is a $B$-Wiener process on $H$. If there exists a $\gamma > 0$ such that*

$$\langle h, Ah\rangle \le -\gamma\|h\|^2 \qquad \forall\, h \in H$$

*then*

$$\limsup_{t\to+\infty}\frac{\|X(t)\|^2}{\log t} \le \frac{e}{\gamma}\,\mathrm{Tr}\, B \text{ a.s.}$$

**Proof.** This is a straightforward generalization of the finite dimensional result, using the exponential martingale inequality; see [27], Chap. II, Theorem 5.5, for the finite dimensional case and [31] for the infinite dimensional generalization. □

Application of this result, together with Sobolev embedding, gives the following corollary which implies (4.4) by the time-reversibility of the stationary OU process.

**Corollary 4.1.** *Assume that $A = \Delta$ on $\mathbb{T}^2$. Let $v = \nabla^\perp\psi$ where $\psi$ is a stationary solution of* (3.1). *If $\sum_{k\in K}\lambda_k|k|^{6+\varepsilon} < +\infty$, for some $\varepsilon > 0$, then $\mathbb{P}$-a.s.*

$$\limsup_{t\to+\infty}\frac{\|v(t)\|_\infty^2}{\log t} \le C < \infty\,.$$

**Proof.** To obtain a bound on the supremum norm in space we use Sobolev embedding. In two dimensions it suffices to bound $v$ in $H^{1+\varepsilon}$, any $\varepsilon > 0$, to get supremum bounds on $v$. We consider (3.4) in the case where $A = \Delta$ on $\mathbb{T}^2$ so that bounding $v$ in $H^{1+\varepsilon}$ is equivalent to bounding $(-A)^{1/2+\varepsilon}v$ in $L^2$. Under the stationary measure $\mathbb{P}$ on $\psi$ we have that each component of

$$(-A)^{\frac{1}{2}+\varepsilon}\nabla^{\perp}\psi \in C(\mathbb{R}, C^2(\mathcal{O})) \subset C(\mathbb{R}, D(A))$$

for some $\varepsilon > 0$. This gives sufficient regularity to the initial data that we can apply Theorem 3.4 (with $s = \frac{1}{2} + \varepsilon$) to show that the two components of $\Theta = (-A)^{1/2+\varepsilon}\nabla^{\perp}\psi$ are strong solutions of (4.5), for appropriate $B$. By (3.5), the two appropriate choices of $B$ are trace class provided that

$$\sum_{k\in K} \lambda_k |k|^{4+\varepsilon} < +\infty$$

which is implied by (4.1). Furthermore we have

$$\langle h, Ah\rangle \le -\alpha_1 \|h\|^2.$$

Thus we have the desired result by application of Theorem 4.2. □

## 5. Numerical Experiment

In this section we conclude with a numerical experiment which illustrates the long-time behavior of the random dynamical system for particle motions and, in particular, demonstrates the existence, and relevance for numerical study of particle distributions, of the random attractor.

We simulate numerically, for 25 time units, the motion of $n = 5000$ particles according to the model (2.2), with $\tau = 1/5$, $\nu = 10^{-2}$ and $\ell = 1/2$, using the Kárman–Obukhov spectrum.[a] We choose the initial conditions for $\psi$ so that $\psi$ is stationary, and the initial conditions for the particle velocities so that the particles are initially at rest, $\dot{x}_i(0) = 0$. We perform two simulations differing only in the initial conditions for the particle positions, $x_i(0)$:

1. On a regular lattice in the *lower-half* of the unit square,

$$x_i(0) = \left(\frac{2k+1}{2m}, \frac{2l+1}{2m}\right), \quad k = 0, \dots, m-1, \quad l = 0, \dots, \frac{m}{2} - 1.$$

2. On a regular lattice in the *upper-half* of the unit square,

$$x_i(0) = \left(\frac{2k+1}{2m}, \frac{2l+1}{2m}\right), \quad k = 0, \dots, m-1, \quad l = \frac{m}{2}, \dots, m-1.$$

Here $m = \sqrt{2n} = 100$. The two simulations use the same realization of $\psi$.

[a] Recall that we cannot prove existence and uniqueness of particle paths in this case. However in practice we do not observe significant mesh effects as we refine the spatial lattice.

Figures 3 and 4 show the final positions of the 5000 particles which started in the lower- and upper-half of the unit square respectively. The particle distributions at later times are virtually indistinguishable, despite the vastly different initial conditions used in the two simulations. This indicates that the particle positions

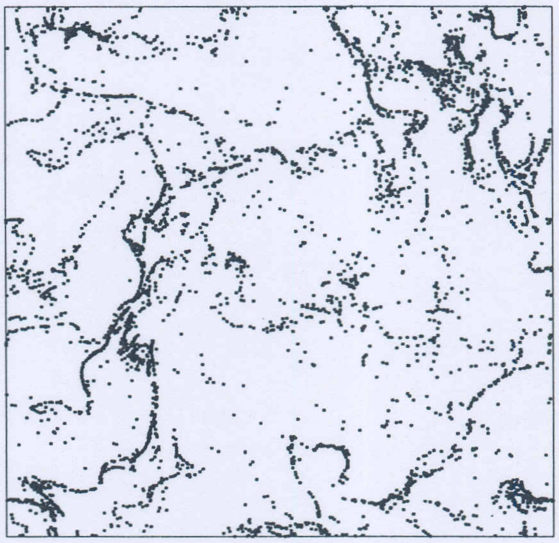

Fig. 3. Final positions of 5000 particles moving for 25 time units in a Kármán-Obukhov velocity field, initially located in the *lower half* of the unit square.

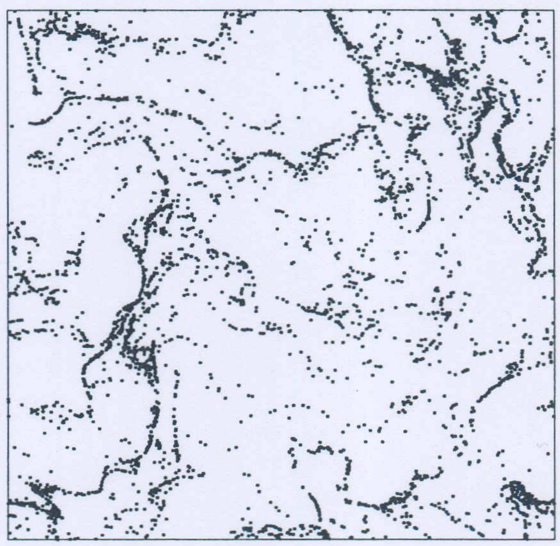

Fig. 4. Final positions of 5000 particles moving for 25 time units in a Kármán-Obukhov velocity field, initially located in the *upper half* of the unit square.

and velocities have converged to a subset of the random attractor; the two figures give a snapshot of the random attractor.

Note also the qualitative agreement between the simulated distributions in Figs. 3 and 4, and the experimental particle distributions in Fig. 1. This illustrates two important facts: (i) that the Gaussian random field model employed here is an effective model for preferential concentration (and this is substantiated further in [32]); (ii) that the random attractor is highly relevant for the study of preferential concentration of inertial particles. Thus further mathematical and numerical study of the model is called for.

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