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008

TRAVELLING COMBUSTION WAVES IN A POROUS MEDIUM. PART II—STABILITY*

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Abstract. The linear stability properties of the travelling combustion waves found in Part I are examined. The key parameters which determine the stability properties of the waves are found to be the (scaled) driving velocity and the solid specific heat. In particular, the destabilising influence of increasing either of these two parameters is demonstrated. The results indicate that travelling combustion waves whose reaction is turned off because the solid temperature becomes too low are always unstable, whereas travelling waves whose reaction is turned off due to depletion of solid reactant can be stable. Global techniques are employed to prove that, for large enough values of the scaled solid specific heat, combustion cannot be sustained in any form, and all initial conditions lead to extinction.

Key words. combustion, travelling waves, stability

AMS(MOS) subject classifications. 35B32, 35B35, 80A30

1. Introduction. In Part I of this paper [7] we demonstrated the existence of steady travelling wave solutions to the simplified model equations governing porous medium combustion derived in [6]. These solutions represent the steady propagation of a combustion zone through combustible solid material. Having constructed these solutions, a natural question of both mathematical interest and physical importance is the one of whether or not these waves are stable. A related matter of interest is the time-dependent behaviour of the governing equations in the region of parameter space where steady solutions do not exist.

The stability of travelling wave solutions to reaction-diffusion equations modelling a form of solid fuel combustion has been examined by Matkowsky and Sivashinsky [5]. Their stability analysis shows how the plane combustion wave loses stability by means of a supercritical Hopf bifurcation as a parameter, proportional to the non-dimensional activation energy, is increased. Thus the existence of a stable periodic travelling wave is demonstrated.

In this paper we analyse the stability of the travelling waves found in [7]. The parameters of importance in determining the stability of the travelling waves are the inlet gas velocity and the specific heat of the combustible solid. Thus, while the magnitude of the activation energy is crucial in enabling us to simplify the nonlinear partial differential equations governing porous medium combustion [6], its role in determining the stability of the travelling combustion waves is a passive one. We demonstrate that travelling combustion waves that possess a reaction rate whose switching mechanism is entirely temperature dependent (a (U, U) switch; see [7]) are always unstable. Travelling combustion waves with a reaction rate whose switching mechanism is determined by exhaustion of solid reactant (a (Q, U) switch; see [7]) are, however, shown to be stable in certain parameter regimes. Thus we show that increasing either the inlet gas velocity or the solid specific heat has a destabilising effect on the plane combustion wave and that the instability is associated with the

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transition between the two distinct forms of travelling wave solution ((U, U) and (Q, U)) defined in [7]. We also prove that for large enough values of the solid specific heat, combustion cannot be sustained in any form and that the ultimate state of the system is the ambient one of no burning, regardless of the initial conditions.

We restrict our attention to the purely one-dimensional model of porous medium combustion, derived in [6]. In practice two-dimensional effects are often important. Nonetheless, the computational results described in [2], [3] suggest that there is a genuine one-dimensional instability of the plane wave and it is this which we analyse.

In §§ 2 and 3 we derive the eigenvalue problems (EVP1 and EVP2) which govern the linear stability of the steady travelling waves for the cases of the (U, U) and (Q, U) switches (defined in [7]) respectively. In §§ 4 and 5 we solve EVP1 and EVP2 in the parameter regime $(\lambda - \lambda_c)$ and $\mu \to 0$. We choose a scaling of the length of the burning zone such that we capture the change from a (U, U) to a (Q, U) switch (that is $L = O(\mu^{1/2})$ as $\mu \to 0$). Finally, in § 6, we perform a global analysis of the time-dependent equations for $\lambda > \lambda_c$, where no steady travelling combustion waves exist.

2. Normal modes analysis—The (U, U) switch case. In this section we derive the eigenvalue problem that governs the (linear) stability of the travelling waves for the case of the (U, U) switch. The time-dependent equations governing porous medium combustion form a moving boundary problem, and it is the difference in the moving boundary conditions for the cases of the (U, U) and the (Q, U) switches that necessitates their separate treatment. For ease of presentation we display the full time-dependent equations derived in [6]. As in [7] we analyse oxygen-rich environments and thus set $a = 0$. The equations are

$$\frac{\partial \sigma}{\partial t} = -\lambda r, \tag{2.1}$$

$$\mu \frac{\partial w}{\partial z} = u - w \tag{2.2}$$

and

$$\sigma \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + w - u + r, \tag{2.3}$$

where

$$r = \mu^{1/2} H(u - u_c) f(w).$$

The boundary conditions are

$$u(\pm\infty, t) = w(-\infty, t) = u_a \quad \text{and} \quad g(-\infty, t) = -1. \tag{2.4}$$

The simple form of the reaction rate r (compare with (1.5) in [7]) is chosen because we are analysing the stability of a (U, U) switch solution, and we consider only small time-dependent perturbations of this solution which keep us within this regime. To perform the stability analysis, we recast the time-dependent equations in a frame fixed with respect to the travelling wave (as in [1]). Thus we make the transformation of independent variables $x = z - ct$ and $t = t$ to (2.1)–(2.4). In addition we define a new variable q by $q = c\sigma$. The resultant equations are

$$\frac{1}{c} \frac{\partial q}{\partial t} - \frac{\partial q}{\partial x} = -\lambda r, \tag{2.5}$$

$$\mu \frac{\partial w}{\partial x} = u - w \tag{2.6}$$

and

$$\frac{q}{c}\frac{\partial u}{\partial t}-q\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2}+w-u+r, \tag{2.7}$$

where r and the boundary conditions are unchanged from above, except that we now have the additional boundary condition on q, namely

$$q(+\infty, t)=c. \tag{2.8}$$

This condition is derived by assuming that no reaction has taken place at $x=+\infty$ since the combustion wave is propagating from left to right.

We define Q, W and U to be the steady solutions of (2.4)–(2.8) and seek perturbations from the basic steady state of the form

$$q(x, t)=Q(x)+e^{cpt}\psi(x),$$
$$w(x, t)=W(x)+e^{cpt}\phi(x)$$

and

$$u(x, t)=U(x)+e^{cpt}\theta(x).$$

We assume that the perturbations are small, relative to the steady solutions, and hence linearise (2.4)–(2.8). This gives

$$\rho\psi-\psi'=-\lambda r_1, \tag{2.9}$$

$$\mu\phi'=\theta-\phi \tag{2.10}$$

and

$$\rho Q\theta-\psi U'-Q\theta'=\theta''+\phi-\theta+r_1, \tag{2.11}$$

where

$$r_1=\mu^{1/2}[H(U-u_c)f_w(W)\phi+\delta(U-u_c)f(W)\theta]. \tag{2.12}$$

Here $'\equiv d/dx$ and henceforth $f_w=df/dw$. The boundary conditions are

$$\psi(\infty)=\phi(-\infty)=\theta(\pm\infty)=0. \tag{2.13}$$

We also require that all solutions are bounded at $x=\pm\infty$.

We have interpreted the derivative of the Heaviside step function as a Dirac delta function in the sense of generalised functions [4]. In effect this allows for the perturbation of the free boundary and yields results identical to those obtained by employing the (more generally applicable) linearisation techniques, which we will use for the case of the (Q, U) switch in the next section.

We now derive the jump conditions which the delta functions impose by integrating (2.9) and (2.11) across the points $x=0$ and $x=1$. We obtain

$$[\psi]_{x=0}=\mu^{1/2}\lambda\frac{f(W(0))\theta(0)}{U'(0)}, \tag{2.14}$$

$$[\psi]_{x=L}=-\mu^{1/2}\lambda\frac{f(W(L))\theta(0)}{U'(L)}, \tag{2.15}$$

$$[\theta']_{x=0}=-[\psi]_{x=0}/\lambda, \tag{2.16}$$

$$[\theta']_{x=L}=-[\psi]_{x=L}/\lambda. \tag{2.17}$$

Having interpreted the delta functions we now redefine r_1 by

(2.12′) $$r_1 = \mu^{1/2} H(U - u_c) f_w(W)\phi.$$

Equations (2.9)–(2.11), (2.12′) and (2.13)–(2.17) constitute an eigenvalue problem for ρ which we denote by EVP1. Since $c > \mu \geqq 0$ we deduce from the form of the perturbation to the steady state, that if $\mathrm{Re}(\rho) > 0$ for *any* part of the spectrum of EVP1 then the travelling wave is unstable. Conversely, if $\mathrm{Re}(\rho) < 0$ for *all* the eigenvalues in the spectrum of EVP1 then we say that the travelling wave is (linearly) stable. However, it may be shown that $\rho \equiv 0$ is an eigenvalue of EVP1 for all values of the parameters. This fact is a direct consequence of the group invariance of the time-dependent equations under the transformation $x \to x + \xi$, for any constant ξ [8]. As such it corresponds to the phase shifts which inevitably occur when a stable travelling wave is perturbed. Since phase shifts do not affect the basic physical properties of the wave, we analyse stability modulo phase-shifts and ignore the zero eigenvalue. In general we cannot say whether or not the spectrum of EVP1 will be discrete since the governing differential equations are posed on the infinite domain.

3. Normal modes analysis—The (Q, U) switch case. We now derive the eigenvalue problem governing the stability of travelling waves in the (Q, U) switch regime. In a moving frame, the time-dependent problem is given by (2.5)–(2.7) where now we require the general form of r,

$$r = \mu^{1/2} H(q - c\tau) H(u - u_c) f(w).$$

The boundary conditions are

$$q(\infty, t) = c \quad \text{and} \quad u(\pm\infty, t) = w(-\infty, t) = u_a.$$

We introduce perturbations from the steady state of the same form as in § 2. The straightforward linearisation of the Heaviside step function performed in the previous section is not applicable for the case of a (Q, U) switch. This is because $q' \equiv 0$ to the left of the burning zone. Thus we must introduce small perturbations to the free boundaries. From the definition of the steady solution in [7], the free boundaries at $x = 0$ and $x = L$ are defined by

(3.1) $$Q(0) = \tau c \quad \text{and} \quad U(L) = u_c.$$

We introduce perturbations to the positions of the free boundaries to obtain

$$q(s_1(t), t) = \tau c \quad \text{and} \quad u(L + s_2(t), t) = u_c$$

where, without loss of generality, we assume that

$$-\infty < s_1 < 0 < L < L + s_2 < \infty.$$

We examine the governing equations in the *small* regions $s_1 < x < 0$ and $L < x < L + s_2$ and derive appropriate jump conditions which represent these small perturbations to the free boundaries. After linearising, there are five zones to consider. We list the five regions and the governing equations within them.

$-\infty < x < s_1$ and $L + s_2 < x < \infty$. In these regions the governing equations are

$$\rho\psi - \psi' = 0,$$

$$\mu\phi' = \theta - \phi$$

and

$$\rho Q\theta - Q\theta' - \psi U' = \theta'' + \phi - \theta.$$

$s_1 < x < 0$ and $L < x < L + s_2$. In these regions the governing equations are

$$\rho\psi - \psi' = -\lambda\mu^{1/2} f(W(x) + e^{c\rho t}\phi(x))\, e^{-c\rho t},$$
$$\mu\phi' = \theta - \phi$$

and

$$\rho Q\theta - Q\theta' - \psi U' = \theta'' + \phi - \theta + \mu^{1/2} f(W(x) + e^{c\rho t}\phi(x))\, e^{-c\rho t}.$$

$O < x < L$. In this region the governing equations are

$$\rho\psi - \psi' = -\lambda\mu^{1/2} f'(W(x))\phi,$$
$$\mu\phi' = \theta - \phi$$

and

$$\rho Q\theta - Q\theta' - \psi U' = \theta'' + \phi - \theta + \mu^{1/2} f'(W(x))\phi.$$

Now, since s_1 and s_2 are assumed to be small (of $O(e^{c\rho t})$), we obtain, by integrating the equation for ψ over $s_1 < x < 0$,

$$-\int_{s_1}^{0} \psi'\, dx = -\int_{s_1}^{0} \lambda\mu^{1/2} f(W(x) + e^{c\rho t}\phi(x))\, e^{-c\rho t}\, dx$$

so that

(3.2) $$\psi(s_1) - \psi(0) \sim \lambda\mu^{1/2} f(W(0))\, e^{-c\rho t} s_1(t).$$

Similarly we obtain

(3.3) $$\theta'(0) - \theta'(s_1) \sim \mu^{1/2} f(W(0))\, e^{-c\rho t} s_1(t),$$

(3.4) $$\psi(L + s_2) - \psi(L) \sim \lambda\mu^{1/2} f(W(L))\, e^{-c\rho t} s_2(t)$$

and

(3.5) $$\theta'(L) - \theta'(L + s_2) \sim \mu^{1/2} f(W(L))\, e^{-c\rho t} s_2(t).$$

The free boundary conditions (3.1) are now linearised. In the neighbourhood of $x = L$ we have

$$U(L) = u_c \quad \text{and} \quad u(L + s_2(t), t) = u_c.$$

Thus, from the form of the perturbation to $U(x, t)$, we obtain

$$U(L + s_2(t)) + e^{c\rho t}\theta(L + s_2(t)) = u_c.$$

Linearising this gives us

$$U'(L) s_2(t) + e^{c\rho t}\theta(L) \sim 0$$

so that

(3.6) $$s_2(t) = -e^{c\rho t}\theta(L)/U'(L).$$

In the neighbourhood of $x = 0$ we have

$$Q(0) = \tau c \quad \text{and} \quad q(s_1(t), t) = \tau c.$$

Thus, from the definition of $q(x, t)$, we obtain

$$Q(s_1(t)) + e^{c\rho t}\psi(s_1(t)) = \tau c.$$

However, $Q(s_1(t)) = \tau c$, since $s_1 < 0$ and since Q is constant outside the burning zone. Thus we have

$$\psi(s_1) = 0. \tag{3.7}$$

Since s_1 and s_2 are assumed small we redefine $x = s_1$ to be $x = 0_-$ and $x = L + s_2$ to be $x = L_+$. Combining (3.2) and (3.3) we eliminate s_1 to obtain the condition

$$[\theta']_{x=0} = -[\psi]_{x=0}/\lambda. \tag{3.8}$$

Eliminating s_2 from (3.4) and (3.5) by use of (3.6) we obtain

$$[\psi]_{x=L} = -\frac{\lambda \mu^{1/2} f(W(L))\theta(L)}{U'(L)} \tag{3.9}$$

and

$$[\theta']_{x=L} = \frac{\mu^{1/2} f(W(L))\theta(L)}{U'(L)}. \tag{3.10}$$

Notice that (3.9) and (3.10) are identical to those derived in the previous section, namely (2.15) and (2.17). This is to be expected since the moving boundary condition is the same at $x = L$ for both the cases of the (U, U) switch and the (Q, U) switch. However, at $x = 0$ the moving boundary condition is different, and thus, for the (Q, U) switch (2.14) is replaced by (3.7).

We now have a second eigenvalue problem for ρ which we denote by EVP2. We have, for $-\infty < x < 0$ and $L < x < \infty$,

$$\rho\psi - \psi' = 0, \tag{3.11}$$

$$\mu\phi' = \theta - \phi \tag{3.12}$$

and

$$\rho Q\theta - Q\theta' - \psi U' = \theta'' + \phi - \theta. \tag{3.13}$$

For $0 < x < L$, we have

$$\rho\psi - \psi' = -\lambda\mu^{1/2} f'(W(x))\phi, \tag{3.14}$$

$$\mu\phi' = \theta - \phi \tag{3.15}$$

and

$$\rho Q\theta - Q\theta' - \psi U' = \theta'' + \phi - \theta + \mu^{1/2} f'(W(x))\phi. \tag{3.16}$$

Equations (3.11)–(3.16) must be solved subject to the jump conditions (3.7)–(3.10) and the boundary conditions

$$\psi(\infty) = \phi(-\infty) = \Theta(\pm\infty) = 0. \tag{3.17}$$

We also require that all components of the solution are bounded at infinity.

As for EVP1, the sign of Re (ρ) determines the stability of the steady travelling combustion wave. A similar argument also shows that $\rho \equiv 0$ is an eigenvalue of EVP2 and again this is a reflection of the fact that it is necessary to examine stability modulo phase-shifts.

4. Solution of EVP1. We solve EVP1 in the parameter regime $(\lambda - \lambda_c)$ and $\mu \to 0$, with the length scaling $L = O(\mu^{1/2})$. We seek eigenvalues which satisfy $\mathrm{Re}(\rho) > 0$. Outside the burning zone we have a linear problem given by

(4.1) $$\rho\psi - \psi' = 0,$$

(4.2) $$\mu\phi' = \theta - \phi$$

and

(4.3) $$\rho Q\theta - Q\theta' - \psi U' = \theta'' + \phi - \theta,$$

where $Q = q_L$ for $-\infty < x < 0$ and $Q = c$ for $L < x < \infty$.

Since (4.1) does not involve θ and ϕ we may solve it explicitly for ψ. Thus we regard the term $\psi U'$ as a forcing function for (4.2) and (4.3) and solve them by means of complementary functions and particular integrals. Equations (4.2) and (4.3) imply that the complementary functions for θ and ϕ are of the form

$$\theta = e^{\xi x} \quad \text{and} \quad \phi = \frac{e^{\xi x}}{1 + \mu\xi}.$$

Substituting these expressions into (4.3) gives

$$\rho Q - Q\xi = \xi^2 - \frac{\mu\xi}{1 + \mu\xi}.$$

Rearranging this gives the following cubic for ξ:

(4.4) $$\mu\xi^3 + (1 + Q\mu)\xi^2 + (Q - \rho Q\mu - \mu)\xi - \rho Q = 0.$$

We now give further details of the solution in the separate regions $-\infty < x < 0$ and $L < x < \infty$.

$-\infty < x < 0$. The solution of (4.1) is

$$\psi = A\, e^{\rho x}$$

which, since we are seeking eigenvalues with $\mathrm{Re}(\rho) > 0$, automatically satisfies the condition that ψ be bounded at $x = -\infty$.

From (3.1) in [7] we know that the steady solution $U(x)$ satisfies

$$U'(x) = \alpha(u_c - u_a)\, e^{\alpha x},$$

where α is found as part of the solution. Thus the particular integral for (4.3) must be the solution of

$$\rho q_L\theta - q_L\theta' - \alpha(u_c - u_a)A\, e^{(\rho+\alpha)x} = \theta'' + \phi - \theta.$$

Hence we seek solutions of the form

$$\theta = M\, e^{(\rho+\alpha)x} \quad \text{and} \quad \phi = \frac{M\, e^{(\rho+\alpha)x}}{1 + \mu(\rho + \alpha)}$$

where M is a constant to be determined. Substitution of θ and ϕ into the governing differential equation gives the equation

(4.5) $$-Mq_L\alpha - (u_c - u_a)\alpha A = M(\rho + \alpha)^2 - \frac{M\mu(\rho + \alpha)}{1 + \mu(\rho + \alpha)}$$

to determine M.

$L < X < \infty$. Since we are seeking eigenvalues which satisfy Re $(\rho) > 0$, and since we require that $\psi(\infty) = 0$, (4.1) gives us

$$\psi(x) = 0.$$

Consequently there is no need to find a particular integral in this domain.

$O < x < L$. As for the steady solution we rescale the length by setting $y = x/\mu^{1/2}$. Furthermore, we anticipate from the form of the expansion of the steady solution in § 5 of [7] that the eigenfunctions ψ, θ and ϕ will be related by

$$\phi = O(\theta) \quad \text{and} \quad \psi = O(\mu^{-1/2}\theta).$$

This is because they satisfy a generalised eigenvalue problem of the form $L\underline{\psi} = \rho B\underline{\psi}$ where L is the Frechet derivative of the nonlinear eigenvalue problem satisfied by the steady solution.

Thus we rescale ψ by introducing a function $\bar{\psi}$, defined by

$$\bar{\psi} = \mu^{1/2}\psi.$$

In addition, the correct scaling for the eigenvalue ρ turns out to be $\rho = O(\mu)$. Thus we expand ρ in the form

$$\rho = \rho_1\mu + \cdots.$$

With these scalings of ψ and ρ, the governing equations (2.9)–(2.11) for ψ, ϕ and θ in $0 < y < L_1$ are

$$\mu^{3/2}\rho_1\bar{\psi} - \bar{\psi}_y = -\lambda\mu^{3/2}f_w(W)\phi, \tag{4.6}$$

$$\mu^{1/2}\phi_y = \theta - \phi \tag{4.7}$$

and

$$\mu^2\rho_1 Q\theta - \bar{\psi}U_y - \mu^{1/2}Q\theta_y = \theta_{yy} + \mu(\phi - \theta) + \mu^{3/2}f_w(W)\phi. \tag{4.8}$$

We solve these equations in series, as $(\lambda - \lambda_c)$ and $\mu \to 0$ subject to the conditions that θ and ϕ are continuous at $x = 0$ and L (that is $y = 0$ and L_1) and that θ' and ψ satisfy the prescribed jump conditions (2.14)–(2.17). Using these continuity and jump conditions we can match the solution from outside the burning zone to the solution inside the burning zone to obtain boundary conditions for (4.6)–(4.8). Note that there is evidence of singular behaviour in (4.7). We circumvent this by solving (4.7) for ϕ as a function of θ and using this exact relationship in the other equations.

Using the known forms of the expansions from [7] and the solution for ψ in $-\infty < x < 0$, we have, from (2.14)

$$\frac{1}{\mu^{1/2}}\bar{\psi}(O_+) - A \sim \frac{1}{\mu^{1/2}} \frac{(\lambda_c - \lambda_{c1}\mu^{3/2})f(u_c + u_{b1}\mu^{3/2})\theta(0)}{U_{1y}(0) + \mu^{3/2}U_{2y}(0)}.$$

Similarly, from (2.15), we obtain

$$\frac{1}{\mu^{1/2}}\bar{\psi}(L_-) \sim \frac{1}{\mu^{1/2}} \frac{(\lambda_c - \lambda_{c1}\mu^{3/2})f(u_c + W_1(L_1)\mu^{3/2})\theta(L_1)}{U_{1y}(L_1) + \mu^{3/2}U_{2y}(L_1)}.$$

It turns out that $A = O(\mu)$, and so we define

$$\bar{A} = \mu^{-1}A.$$

Then, for ease of notation, we write these conditions on $\bar{\psi}$ as

$$\bar{\psi}(O_+) \sim (\lambda_c - \lambda_{c1}\mu^{3/2})(K_1 + K_2\mu^{3/2})\theta(0) + \mu^{3/2}\bar{A} \tag{4.9}$$

and

(4.10) $$\bar{\psi}(L_-) \sim (\lambda_c - \lambda_{c1}\mu^{3/2})(J_1 + J_2\mu^{3/2})\theta(L_1),$$

where K_1, K_2, J_1 and J_2 are determined in an obvious way from the previous equations.

We now find the approximate form of the solutions for θ and ϕ outside the burning zone so that we can derive the matching conditions on these functions at $x=0$ and L. To first order in μ, expression (4.5) gives

$$M = g(\rho_1)\bar{A},$$

where

(4.11) $$g(\rho_1) = \frac{(u_c - u_a)\alpha_1}{[\rho_1 - \rho_1^2 - 2\rho_1\alpha_1]}.$$

Here we have used the fact that $q_L \sim Q_1(0)\mu = (1-\alpha_1)\mu$ (see (5.15) in [7].)

Notice that, for $\rho_1 = 0$ and $\rho_1 = 1 - 2\alpha_1$, $g(\rho_1)$ is undefined. This is because, for these values of ρ_1, the exponent in the forcing term $\psi U'$ in (4.3) is equal to one of the roots of the characteristic cubic (4.4). Hence we require particular integrals of the form $x\, e^{(\rho+\alpha)x}$ to examine the cases $\rho_1 = 0$ and $\rho_1 = 1 - 2\alpha_1$.

Analysis of the cubic (4.4) shows that in $-\infty < x < 0$ the only positive root is given, to first order in $\mu \to 0$, by $\xi \sim \xi_1\mu$, where

(4.12) $$\xi_1 = \frac{\alpha_1 + [\alpha_1^2 + 4\rho_1(1-\alpha_1)]^{1/2}}{2}.$$

Thus the solution for θ and ϕ in $-\infty < x < 0$ is given, to first order, by

$$\theta(x) = A_1 \exp(\xi_1\mu x) + g(\rho_1)\bar{A} \exp\{(\rho_1 + \alpha_1)\mu x\}$$

and

$$\phi(x) = \frac{A_1 \exp(\xi_1\mu x)}{1 + \xi_1\mu^2} + \frac{g(\rho_1)\bar{A} \exp\{(\rho_1 + \alpha_1)\mu x\}}{1 + (\rho_1 + \alpha_1)\mu^2}.$$

Analysis of the cubic (4.4) shows that in $L < x < \infty$ the two negative roots $\mu\xi_2$ and $\mu\xi_3$ are determined by

(4.13) $$\xi_2 \sim -1/\mu^2 - 1$$

and

(4.14) $$\xi_3 \sim \frac{1 - c_1 - [(c_1 - 1)^2 + 4\rho_1 c_1]^{1/2}}{2}$$

where $c \sim c_1\mu$. Here $c_1 = Q_1(L_1)$, since the wave speed c is determined by $c = Q(L)$. Thus the solution in $-\infty < x < 0$ is given, to first order, by

$$\theta(x) = \sum_{i=2}^{3} A_i \exp\{\xi_i\mu(x - L)\}$$

and

$$\phi(x) = \sum_{i=2}^{3} \frac{A_i \exp\{\xi_i\mu(x - L)\}}{1 + \mu^2\xi_i}.$$

Using these expressions for θ and ϕ outside the burning zone we show, by continuity and application of the jump conditions (2.16) and (2.17), that in the rescaled variable y

(4.15) $$\theta(O_+) = A_1 + g(\rho_1)\bar{A},$$

(4.16) $$\phi(O_+) = \frac{A_1}{1+\xi_1\mu^2} + \frac{g(\rho_1)\bar{A}}{1+(\rho_1+\alpha_1)\mu^2},$$

(4.17) $$\theta(L_-) = A_2 + A_3,$$

(4.18) $$\phi(L_-) = \frac{A_2}{1+\mu^2\xi_2} + \frac{A_3}{1+\mu^2\xi_3},$$

(4.19) $$\theta_y(O+) = -(K_1 + K_2\mu^{3/2})\theta(0) + \mu^{3/2}A_1\xi_1 + \mu^{3/2}\bar{A}(\rho_1+\alpha_1)g(\rho_1)$$

and

(4.20) $$\theta_y(L_-) = -(J_1 + J_2\mu^{3/2})\theta(L) + \mu^{3/2}A_2\xi_2 + \mu^{3/2}A_3\xi_3$$

where K_1, K_2, J_1 and J_2 are defined as for the jump conditions on $\bar{\psi}$.

In summary we must now solve (4.6)–(4.8) subject to (4.9), (4.10) and (4.15)–(4.20). We seek a series solution of this problem in powers of $\mu^{1/2}$ and expand $\bar{\psi}$, θ and ϕ as

$$\bar{\psi} \sim \bar{\psi}_1 + \mu^{3/2}\bar{\psi}_2,$$
$$\theta \sim \theta_1 + \mu^{3/2}\theta_2$$

and

$$\phi \sim \phi_1 + \mu^{1/2}\phi_2.$$

To first order we obtain the general solution

$$\bar{\psi}_1 = B \quad \text{and} \quad \theta_1 = \phi_1 = Ey + F$$

where B, E and F are constants of integration.

The precise form of the expression for ϕ now becomes clear: substituting θ_1 into the linear equation (4.7), applying (4.12) and integrating, we obtain

$$\phi(y) \sim \exp\{-\mu^{-1/2}y\}[A_1 + g(\rho_1)\bar{A}] + Ey + F$$
$$- \exp\{-\mu^{-1/2}y\}F - \mu^{1/2}E[1 - \exp\{-\mu^{-1/2}y\}].$$

Thus, as for the steady solution (5.12) in [7], we see the effect of the boundary layer caused by the singularly perturbed nature of the equation for ϕ. Using this expression for ϕ to avoid the necessity of a full boundary layer analysis in the neighbourhood of $y=0$, we calculate $\bar{\psi}$ and θ to second order. We obtain

$$\bar{\psi}_2 = D + \rho_1 By + \lambda_c f_w(u_c)\left(\frac{Ey^2}{2} + Fy\right) + O(\mu^{1/2})$$

and

$$\theta_2 = Gy + H + p_1(y)E + p_2(y)B - p_3(y)F.$$

Here D, G and H are constants of integration. The polynomials $p_i(y)$ are defined by

(4.21) $$p_1(y) = \frac{\alpha_1 y^2}{2} - [\lambda_c f(u_c) + f_w(u_c)]\frac{y^3}{6},$$

(4.22) $$p_2(y) = f(u_c)\frac{y^3}{6} - (u_c - u_a)\alpha_1\frac{y^2}{2}$$

and

(4.23) $$p_3(y)=f_w(u_c)\frac{y^2}{2}.$$

Having solved (4.6)–(4.8) to second order we now match the solutions at $x=0$ and L. Conditions (4.9), (4.10), (4.15) and (4.17)–(4.20) give, respectively,

$$B+\mu^{3/2}D\sim\mu^{3/2}\bar{A}+(\lambda_c-\lambda_{c1}\mu^{3/2})(K_1+K_2\mu^{3/2})(F+H\mu^{3/2}),$$

$$B+\mu^{3/2}\left[D+\rho_1L_1B+\lambda_cf_w(u_c)\left(\frac{EL_1^2}{2}+FL_1\right)\right]$$
$$\sim(\lambda_c-\lambda_{c1}\mu^{3/2})(J_1+J_2\mu^{3/2})$$
$$\cdot\{EL_1+F+\mu^{3/2}[GL_1+H+p_1(L_1)E+p_2(L_1)B-p_3(L_1)F]\},$$

$$F\sim A_1+g(\rho_1)\bar{A}+O(\mu^{3/2}),$$

(4.24) $$EL_1+F\sim A_2+A_3+O(\mu^{3/2}),$$

(4.25) $$EL_1+F-\mu^{1/2}E\sim\frac{A_2}{1+\xi_2\mu^2}+\frac{A_3}{1+\xi_3\mu^2}+O(\mu^{3/2}),$$

$$E+\mu^{3/2}G\sim\mu^{3/2}[A_1\xi_1+\bar{A}(\rho_1+\alpha_1)g(\rho_1)]-[K_1+K_2\mu^{3/2}][F+H\mu^{3/2}]$$

and

$$E+\mu^{3/2}G+\{p_{1y}(L_1)E+p_{2y}(L_1)B-p_{3y}(L_1)F\}$$
$$\sim\mu^{3/2}[A_2\xi_2+A_3\xi_3]-[J_1+J_2\mu^{3/2}]$$
$$\cdot(EL_1+F+\mu^{3/2}[GL_1+H+p_1(L_1)E+p_2(L_1)B-p_3(L_1)]).$$

Notice that since $1+\mu^2\xi_2\sim-\mu^2$ (from (4.13)), (4.24) and (4.25) imply that

$$A_2\sim\mu^{5/2}E.$$

Matching θ to $O(1)$ and θ_y and $\bar{\psi}$ to $O(\mu^{3/2})$ we obtain

(4.26) $$B=\lambda_cK_1F,$$

(4.27) $$D=\bar{A}+\lambda_cK_1H+\lambda_cK_2F-\lambda_{c1}K_1F,$$

(4.28) $$B=\lambda_cJ_1(EL_1+F),$$

$$D+\rho_1L_1B+\lambda_cf_w(u_c)\left(\frac{EL_1^2}{2}+FL_1\right)$$

(4.29) $$=(\lambda_cJ_2-\lambda_{c1}J_1)(EL_1+F)+\lambda_cJ_1$$
$$\cdot\{GL_1+H+p_1(L_1)E+p_2(L_1)B-p_3(L_1)F\},$$

(4.30) $$F=A_1+g(\rho_1)\bar{A},$$

(4.31) $$EL_1+F=A_3,$$

(4.32) $$E=-K_1F,$$

(4.33) $$G=A_1\xi_1+\bar{A}(\rho_1+\alpha_1)g(\rho_1)-K_1H-K_2F,$$

(4.34) $$E=-J_1(EL_1+F)$$

and

$$
\begin{aligned}
&G+p_{1y}(L_1)E+p_{2y}(L_1)B-p_{3y}(L_1)F \\
&\quad =A_3\xi_3-J_2[EL_1+F] \\
&\qquad -J_1[GL_1+H+p_1(L_1)E+p_2(L_1)B-p_3(L_1)F].
\end{aligned} \tag{4.35}
$$

Thus we have ten equations in the nine unknowns $\bar{A}$, A_1, A_3, B, D, E, F, G and H. However, by use of (4.26) and (4.28), respectively, it can be shown that (4.32) and (4.34) are not linearly independent and both imply that

$$E=-B/\lambda_c. \tag{4.36}$$

The eigenvalues ρ_1 will be determined by making the system of nine equations (4.26)-(4.31), (4.33), (4.35) and (4.36) singular. We now investigate for which values of ρ_1 this occurs.

Consider the three equations (4.26), (4.28) and (4.36) which only involve B, E and F. They may be written as

$$
\begin{pmatrix} 1 & 0 & -\lambda_c K_1 \\ 1 & -\lambda_c J_1 L_1 & -\lambda_c J_1 \\ 1 & \lambda_c & 0 \end{pmatrix}\begin{pmatrix} B \\ E \\ F \end{pmatrix}=\underset{\sim}{0}. \tag{4.37}
$$

The determinant Δ of this matrix system is

$$
\begin{aligned}
\Delta &= \lambda_c^2 J_1-\lambda_c K_1[\lambda_c+\lambda_c J_1 L_1] \\
&= \lambda_c^2[J_1-K_1(1+J_1L_1)].
\end{aligned} \tag{4.38}
$$

From the implicit definitions of K_1 and J_1 ((4.9) and (4.10)) we have

$$K_1=\frac{f(u_c)}{U_{1y}(0)} \quad \text{and} \quad J_1=\frac{f(u_c)}{U_{1y}(L_1)}.$$

Equations (5.16) and (5.17) in [7] show us that

$$U_1(y)=\frac{f(u_c)}{2}(L_1y-y^2).$$

Hence

$$K_1=\frac{2}{L_1} \quad \text{and} \quad J_1=-\frac{2}{L_1}.$$

This implies that

$$J_1=K_1(1+J_1L_1)$$

and hence, by (4.38), the determinant Δ of the 3×3 matrix system (4.37), which determines B, E and F, is zero.

Thus we may solve (4.37) for the family of eigenvectors $(B, E, F)=(1, -\lambda_c^{-1}, (\lambda_c^{-1}/2)L_1)B^*$, where B^* is any complex number. By (4.31) we obtain $A_3=(1/2)\lambda_c^{-1}L_1B^*$. If we substitute these known values of B, E, F and A_3 into (4.30), (4.33), (4.27), (4.29) and (4.35), respectively, we obtain a matrix system of the form

$$
\begin{pmatrix} 1 & g(\rho_1) & 0 & 0 & 0 \\ \xi_1 & (\rho_1+\alpha_1)g(\rho_1) & -1 & 0 & -K_1 \\ 0 & 1 & 0 & -1 & \lambda_c K_1 \\ 0 & 0 & \lambda_c J_1 L_1 & -1 & \lambda_c J_1 \\ 0 & 0 & 1+J_1L_1 & 0 & J_1 \end{pmatrix}\begin{pmatrix} A_1 \\ \bar{A} \\ G \\ D \\ H \end{pmatrix}=\underset{\sim}{r}B^*
$$

where $\underline{r}$ is a known vector whose components are linear combinations of B, E, F and A_3.

By use of elementary row operations we can reduce this matrix system to the equivalent system

$$(4.39)\qquad \begin{pmatrix} 1 & g(\rho_1) & 0 & 0 & 0 \\ \xi_1 & (\rho_1+\alpha_1)g(\rho_1)+\lambda_c^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & \lambda_c K_1 \\ 0 & 0 & \lambda_c J_1 L_1 & -1 & \lambda_c J_1 \\ 0 & 0 & 1 & \lambda_c^{-1} & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ \bar{A} \\ G \\ D \\ H \end{pmatrix} = \underline{r}' B^*.$$

Analysis of the determinant of this system shows that it is

$$[(\rho_1+\alpha_1)g(\rho_1)+\lambda_c^{-1}-\xi_1 g(\rho_1)]\bar{\Delta}$$

where $\bar{\Delta}$ is the determinant of the 3×3 sub-system of equations for G, D and H. However

$$\bar{\Delta}=\lambda_c[K_1(1+J_1L_1)-J_1]=-\frac{\Delta}{\lambda_c}.$$

Since $\Delta=0$ we have $\bar{\Delta}=0$ and hence the system of (4.39) is singular.

Thus the discrete spectrum of the eigenvalue ρ_1, with $\operatorname{Re}(\rho_1)>0$, will be determined by those values of ρ_1 for which the vector $\underline{r}'$ is in the range of the singular 5×5 matrix defined by (4.39).

However, for $B^*=0$, the two equations for A_1 and $\bar{A}$ may be solved uniquely to give $A_1=\bar{A}=0$, provided that $\rho_1\neq 0$ and $\rho_1\neq 1-2\alpha_1$. Thus, in this case, (4.39) reduce to

$$(4.40)\qquad \begin{pmatrix} 0 & -1 & \lambda_c K_1 \\ \lambda_c J_1 L_1 & -1 & \lambda_c J_1 \\ 1 & \lambda_c^{-1} & 0 \end{pmatrix} \begin{pmatrix} G \\ D \\ H \end{pmatrix} = \underline{0}.$$

Because $\bar{\Delta}=0$, (4.40) possess a nontrivial eigenvector $(G, D, H)=(-\lambda_c^{-1}, 1, (1/2)\lambda_c^{-1}L_1)$. Since this eigenvector exists for all values of ρ_1, we have the following theorem.

THEOREM 4.1. *In the parameter regime* $(\lambda-\lambda_c)$ *and* $\mu\to 0$, *with the length scaling of* $L=O(\mu^{1/2})$, *the travelling wave corresponding to a* (U, U) *switch is unstable.*

Proof. We have demonstrated above that the spectrum of the generalised eigenvalue problem (2.18), governing the stability of the (U, U) switch travelling wave, includes a continuous part comprising eigenvalues of the form $\rho\sim\rho_1\mu$ where ρ_1 can be any complex number satisfying $\operatorname{Re}(\rho_1)>0$, provided that $\rho_1\neq 1-2\alpha_1$. Consequently the steady travelling wave solution is unstable.

5. Solution of EVP2. We solve EVP2 in the parameter regime $(\lambda-\lambda_c)$ and $\mu\to 0$ and $L=O(\mu^{1/2})$. The analysis is very similar to that in the previous section where we solved EVP1. The differences arise only from the form of the steady-state solution and from the (different) moving boundary condition at $x=0$. The form of the series expansions of the eigenfunctions and the eigenvalue remain unaltered. Thus we describe only the differences between the two problems and then proceed directly to the matching conditions at $x=0$ and L.

$-\infty<x<0$. In the case of EVP2 we have a moving boundary condition of the form (3.7). That is

$$\psi(O_-)\sim\psi(s_1)=0.$$

Hence $\psi \equiv 0$ for $-\infty < x < 0$. This means that there is no particular integral required for the solution of the equations for θ and ϕ. The cubic (4.4), which determines the complementary functions, has only one positive root $\xi \sim \xi_1 \mu$, where

$$2\xi_1 = (1-\tau c_1) + [(1-\tau c_1)^2 + 4\rho_1 \tau c_1]^{1/2}. \tag{5.1}$$

Here c_1 is the first term in the series expansion of the wave-speed c and is given by $c_1 = Q_1(L_1)$. Thus the solutions for θ and ϕ in $-\infty < x < 0$ are given by

$$\theta(x) = A_1 \exp\{\xi_1 \mu x\}$$

and

$$\phi(x) = \frac{A_1 \exp\{\xi_1 \mu x\}}{1+\xi_1 \mu^2}.$$

$0 < x < L$. In $0 < x < L$ the only change from the solution of EVP1 is that the number α_1, upon which they depend, is determined by the series expansion of the (Q, U) switch solution given in [7].

$L < x < \infty$. The solution in this regime is completely unchanged. Again the number c_1, upon which the roots of the cubic (6.4.4) depend, is determined from the series expansion of the (Q, U) switch solution given in [7].

The matching is similar to that for the (U, U) switch except that rather than having individual jump conditions for $\bar{\psi}(0)$ and $\theta_y(0)$ we have only the one condition (2.16). This is compensated for by the fact that, for $x < 0$, $\bar{\psi} \equiv 0$. Matching $\bar{\psi}_y(L)$, $\theta(0)$, $\theta(L)$, $\phi(L)$, $\theta_y(0)$ and $\theta_y(L)$ respectively, we obtain

$$\begin{aligned} &B + \mu^{3/2}\left[D + \rho_1 L_1 B + \lambda_c f_w(u_c)\left(\frac{EL_1^2}{2} + FL_1\right)\right] \\ &\quad \sim (\lambda_c - \lambda_{c1}\mu^{3/2})(J_1 + J_2\mu^{3/2}) \\ &\qquad \cdot \{EL_1 + F + \mu^{3/2}[GL_1 + H + p_1(L_1)E + p_2(L_1)B - p_3(L_1)F]\}, \end{aligned}$$

$$A_1 \sim F + \mu^{3/2}H,$$

$$\begin{cases} A_2 + A_3 \sim EL_1 + F, \\ \dfrac{A_2}{1+\mu^2\xi_2} + \dfrac{A_3}{1+\mu^2\xi_3} \sim EL_1 + F - \mu^{1/2}E, \end{cases} \tag{5.2}$$

$$E + \mu^{3/2}G \sim \mu^{3/2}A_1\xi_1 - \frac{[B + D\mu^{3/2}]}{\lambda_c - \lambda_{c1}\mu^{3/2}},$$

and

$$\begin{aligned} &E + \mu^{3/2}G + \mu^{3/2}\{p_{1y}(L_1)E + p_{2y}(L_1)B - p_{3y}(L_1)F\} \\ &\quad \sim (A_2\xi_2 + A_3\xi_3)\mu^{3/2} - [J_1 + J_2\mu^{3/2}] \\ &\qquad \cdot \{EL_1 + F + \mu^{3/2}[GL_1 + H + p_1(L_1)E + p_2(L_1)B - p_3(L_1)F]\}. \end{aligned}$$

Similarly, as in the previous section (5.2) demonstrate that

$$A_2 \sim -\mu^{5/2}(c_1 - 1)E.$$

Thus, to the orders of magnitude in which we are interested, A_2 decouples from the equations.

Taking these equations in turn and equating the conditions on $\psi(L)$, $\theta_y(0)$, $\theta_y(L)$ and $\theta(0)$ to $O(\mu^{3/2})$ and the condition on $\theta(L)$ *to* 0(1) (since to $O(\mu^{3/2})$ the matching condition on $\theta(L)$ involves the second term in the series expansion of ρ), we obtain

$$B=\lambda_c J_1(EL_1+F), \tag{5.3}$$

$$\begin{aligned} D+\rho_1 L_1 B+\lambda_c f_w(u_c)\left(\frac{EL_1^2}{2}+FL_1\right) &= (\lambda_c J_2-\lambda_{c1}J_1)(EL_1+F) \\ &\quad +\lambda_c J_1[p_1(L_1)E+p_2(L_1)B-p_3(L_1)F+GL_1+H], \end{aligned} \tag{5.4}$$

$$A_1=F, \tag{5.5}$$

$$H=0, \tag{5.6}$$

$$A_3=EL_1+F, \tag{5.7}$$

$$E=-B/\lambda_c, \tag{5.8}$$

$$G=A_1\xi_1-\frac{D}{\lambda_c}-\frac{B\lambda_{c1}}{\lambda_c^2}, \tag{5.9}$$

$$E=-J_1(EL_1+F) \tag{5.10}$$

and

$$\begin{aligned} G+p_{1y}(L_1)E+p_{2y}(L_1)B-p_{3y}(L_1)F &= A_3\xi_3-J_2(EL_1+F) \\ &\quad -J_1[p_1(L_1)E+p_2(L_1)B-p_3(L_1)F+GL_1+H]. \end{aligned} \tag{5.11}$$

Thus we have nine equations in the eight unknowns A_1, A_3, B, D, E, F, G and H. However, as in the last section, we find that two equations are linearly dependent—in this case (5.3) and (5.10) combine to give (5.8). Thus we have a set of eight equations in eight unknowns. The eigenvalues ρ_1 are determined by those values of ρ_1 for which the system of eight equations is singular. We now examine this further.

We eliminate A_1, A_3, E, G and H to obtain three equations for B, D and F. Eliminating E between (5.3) and (5.8) gives

$$B=\left(\frac{\lambda_c J_1}{1+J_1L_1}\right)F. \tag{5.12}$$

Eliminating E, EL_1+F, G and H using (5.8), (5.3), (5.9), (5.5) and (5.6) in (5.4) gives us

$$\begin{aligned} D+\rho_1L_1B-f_w(u_c)L_1^2B/2+\lambda_c f_w(u_c)L_1F &= (\lambda_c J_1-\lambda_{c1}J_1)\frac{B}{\lambda_c J_1}+\lambda_c J_1 p_1(L_1)\left(-\frac{B}{\lambda_c}\right) \\ &\quad +\lambda_c J_1 p_2(L_1)B-\lambda_c J_1 p_3(L_1)F+\lambda_c J_1\left[F\xi_1-\frac{D}{\lambda_c}-\frac{B\lambda_{c1}}{\lambda_c^2}\right]. \end{aligned} \tag{5.13}$$

Combining (5.3), (5.4) and (5.11) we obtain

$$\begin{aligned} G+p_{1y}(L_1)E+p_{2y}(L_1)B-p_{3y}(L_1)F &= A_3\xi_3-\lambda_c^{-1}\left\{D+\rho_1L_1B+\lambda_c f_w(u_c)\left[\frac{EL_1^2}{2}+FL_1\right]\right\}-\frac{B\lambda_{c1}}{\lambda_c^2}. \end{aligned}$$

Eliminating G, E and A_3 from this expression, by use of (5.9), (5.8) and (5.7), gives us

$$p_{1y}(L_1)\left(-\frac{B}{\lambda_c}\right)+p_{2y}(L_1)B-p_{3y}(L_1)F$$

(5.14)

$$=\xi_3\left(\frac{B}{\lambda_c J_1}\right)-\frac{\rho_1 L_1 B}{\lambda_c}+\frac{f_w(u_c)L_1^2 B}{2\lambda_c}-f_w(u_c)L_1F-\xi_1 F.$$

Equations (5.12)-(5.14) may be written more succinctly as

$$\begin{pmatrix} 1 & 0 & -a \\ k_1(\rho_1) & 1 & k_2(\rho_1) \\ k_3(\rho_1) & 0 & k_4(\rho_1) \end{pmatrix}\begin{pmatrix} B \\ D \\ F \end{pmatrix}=\underline{0}$$

where

$$a=\frac{\lambda_c J_1}{1+J_1L_1},$$

$$k_3(\rho_1)=\frac{\xi_3}{\lambda_c J_1}-\frac{\rho_1 L_1}{\lambda_c}+\frac{f_w(u_c)L_1^2}{2\lambda_c}+\frac{p_{1y}(L_1)}{\lambda_c}-p_{2y}(L_1),$$

$$k_4(\rho_1)=-\xi_1-f_w(u_c)L_1+p_{3y}(L_1)$$

and $k_1(\rho_1)$ and $k_2(\rho_2)$ are determined by (5.13). The determinant of the 3×3 matrix described above is singular if and only if

$$k_4(\rho_1)+ak_3(\rho_1)=0.$$

Consequently the eigenvalues ρ_1 are determined by the eigenrelation

$$(1+J_1L_1)[\xi_1+f_w(u_c)L_1-p_{3y}(L_1)]$$
$$=\lambda_c J_1\left[\frac{\xi_3}{\lambda_c J_1}-\frac{\rho_1 L_1}{\lambda_c}+\frac{f_w(u_c)L_1^2}{2\lambda_c}+\frac{p_{1y}(L_1)}{\lambda_c}-p_{2y}(L_1)\right].$$

From (4.23) we have

$$p_{3y}(L_1)=f_w(u_c)L_1.$$

Also, using (4.21) and (4.22) we deduce that

$$\frac{f_w(u_c)L_1^2}{2}+p_{1y}(L_1)-\lambda_c p_{2y}(L_1)=2\alpha_1 L_1-\lambda_c f(u_c)L_1^2.$$

Thus the eigenrelation becomes

$$(1+J_1L_1)\xi_1=\xi_3-L_1J_1\rho_1+L_1J_1(2\alpha_1-\lambda_c f(u_c)L_1).$$

From the implicit definition of J_1 in (4.10) and from the derivation of $U_y(L_1)$, for the (Q, U) switch solution in § 5 of [7], we deduce that

$$J_1=\frac{f(u_c)}{U_{1y}(L_1)}=\frac{\lambda_c f(u_c)}{\alpha_1-\lambda_c f(u_c)L_1}.$$

Thus

$$1+J_1L_1=\frac{\alpha_1}{\alpha_1-\lambda_c f(u_c)L_1}.$$

Since $c_1 = Q_1(L_1)$ we have, from the solution of the steady problem given in § 5 of [7],

$$c_1 = 1 - \alpha_1 + \lambda_c f(u_c) L_1.$$

Using (5.10) in [7] and the free boundary condition (5.6) in [7], respectively, we obtain

$$\alpha_1 = 1 - \frac{\tau \lambda_c f(u_c) L_1}{1-\tau} \quad \text{and} \quad \alpha_1 = 1 - \tau c_1.$$

Thus

$$c_1 = \frac{\lambda_c f(u_c) L_1}{1-\tau}. \tag{5.15}$$

Hence the eigenrelation for ρ_1 may be written as

$$(1-\tau c_1)\xi_1 = (1-c_1)\xi_3 - c_1(1-\tau)\rho_1 + c_1(1-\tau)[2-(1+\tau)c_1].$$

Here ξ_1 and ξ_3 are functions of ρ_1 defined by (5.1) and (4.14). The solution of this eigenrelation may be reduced to the problem of determining the roots of a quartic. The details may be found in [9]. The roots are found to be $\rho_1 = 0$ (twice) and, for

$$c_1 > \frac{1+\tau}{2\tau},$$

a further positive root

$$\rho_1 = \frac{[2\tau c_1 - (1+\tau)][(1+\tau)c_1 - 2]}{c_1(1-\tau)^2}.$$

By (5.15) we deduce the existence of a positive eigenvalue ρ_1, whenever

$$L_1 \geqq \left(\frac{1+\tau}{2\tau}\right)\left(\frac{1-\tau}{\lambda_c f(u_c)}\right).$$

We know that EVP2 admits an eigenvalue $\rho \equiv 0$. If we now assume that this eigenvalue has algebraic multiplicity two, as indicated by the series expansion, then we obtain the following result.

THEOREM 5.1. *Assume that the eigenvalue $\rho = 0$ of* EVP2 *has algebraic multiplicity two. Then, in the parameter regime $(\lambda - \lambda_c)$ and $\mu \to 0$, with the length scaling $L = L_1 \mu^{1/2}$, there exists a critical value of the length L_1, denoted by L_c, such that for $L_1 < L_c$ the travelling wave corresponding to a (Q, U) switch is (linearly) stable, while for $L_1 > L_c$ it is unstable.*

The critical value is

$$L_c = \left(\frac{1+\tau}{2\tau}\right)\left(\frac{1-\tau}{\lambda_c f(u_c)}\right).$$

Proof. We have demonstrated above that the spectrum of the eigenvalue problem governing the stability of the (Q, U) switch travelling wave has eigenvalues of the form $\rho \sim \rho_1 \mu$. Furthermore, we have shown that for $L_1 < L_c$ there are no eigenvalues ρ_1 with positive real part, while for $L_1 > L_c$ there is one eigenvalue ρ_1 with positive real part.

By Theorem 5.2 in [7] it may be shown that L_c, the critical value L_1, lies in the range of existence of (Q, U) switch solutions. Hence the result follows. □

6. Global analysis for specific heat above critical. In this section we analyse the time-dependent equations (2.1)–(2.3) when the specific heat, λ, is greater than the critical value, λ_c. In this case steady travelling wave solutions do not exist (see Theorem 5.1 in [7]) and we prove that the ultimate state of the system is the ambient state of no burning. This information is valuable because it demonstrates that there are no stable periodic travelling wave solutions for $\lambda \geqq 2\lambda_c$.

In order that the various integrals we require are defined, we normalise u and v, with respect to the ambient temperature, by setting $\hat{u} = u - u_a$ and $\hat{w} = w - u_a$. We also generalise the reaction rate slightly to the case where f is a general *positive* function of $\hat{u}$, $\hat{v}$ and σ and define

$$\hat{r} = \mu^{1/2} H(\hat{u} + u_a - u_c) H(\sigma - \tau c) f(\hat{u}, \hat{w}, \sigma).$$

For convenience of notation we now define the weighted L_2-norm of a function F to be

$$\|F\|_\omega = \left(\int_{-\infty}^{+\infty} \omega(z) F(z)^2 \, dz \right)^{1/2}$$

where $\omega(z)$ is the (positive) weight function.

THEOREM 6.1. *For $\lambda \geqq 2\lambda_c$ the system of equations* (2.1)–(2.3) *subject to the boundary conditions* (2.4) *and arbitrary initial conditions satisfies $\hat{u}(z, t) \to 0$ as $t \to \infty$.*

Proof. Multiply (2.3) by $\hat{u}$ and integrate with respect to z from $-\infty$ to $+\infty$. If we use (2.2) to eliminate $\hat{w} - \hat{u}$ then we obtain

$$\int_{-\infty}^{+\infty} \frac{\sigma}{2} \frac{\partial}{\partial t}(\hat{u}^2) \, dz = \int_{-\infty}^{+\infty} \hat{u} \frac{\partial^2 \hat{u}}{\partial z^2} \, dz - \int_{-\infty}^{+\infty} \mu \hat{u} \frac{\partial \hat{w}}{\partial z} \, dz + \int_{-\infty}^{+\infty} \hat{u} \hat{r} \, dz. \tag{6.1}$$

Integrating by parts, applying the boundary conditions and noting that

$$\sigma \frac{\partial}{\partial t}(\hat{u}^2) = \frac{\partial}{\partial t}(\sigma \hat{u}^2) - \frac{\partial \sigma}{\partial t} \hat{u}^2,$$

(6.1) gives us

$$\frac{1}{2} \frac{\partial}{\partial t}(\|\hat{u}\|_\sigma^2) = -\left\| \frac{\partial \hat{u}}{\partial z} \right\|_1^2 - \int_{-\infty}^{+\infty} \mu \hat{u} \frac{\partial \hat{w}}{\partial z} \, dz + \int_{-\infty}^{+\infty} \hat{r} \left(\hat{u} - \lambda \frac{\hat{u}^2}{2} \right) dz. \tag{6.2}$$

If we differentiate (2.2) with respect to z, multiply through by $\hat{w}$ and integrate with respect to z, then we obtain

$$\mu \int_{-\infty}^{+\infty} \hat{w} \frac{\partial^2 \hat{w}}{\partial z^2} \, dz = \int_{-\infty}^{+\infty} \hat{w} \frac{\partial \hat{u}}{\partial z} \, dz - \int_{-\infty}^{+\infty} \frac{1}{2} \frac{\partial}{\partial z} (\hat{w})^2 \, dz. \tag{6.3}$$

Taking the boundary condition $\hat{u}(\infty, t) = 0$ in conjunction with (2.3) gives $\hat{w}(\infty, t) = 0$. Thus, integrating (6.3) by parts, we have

$$-\mu \int_{-\infty}^{+\infty} \left(\frac{\partial \hat{w}}{\partial z} \right)^2 dz = -\int_{-\infty}^{+\infty} \hat{u} \frac{\partial \hat{w}}{\partial z} \, dz.$$

Substituting this into (6.2) yields

$$\frac{1}{2} \frac{\partial}{\partial t}(\|\hat{u}\|_\sigma^2) = -\left\| \frac{\partial \hat{u}}{\partial z} \right\|_1^2 - \mu^2 \left\| \frac{\partial \hat{w}}{\partial z} \right\|_1^2 + \int_{-\infty}^{+\infty} \hat{r} \left(\hat{u} - \frac{\lambda \hat{u}^2}{2} \right) dz.$$

Since $\hat{u} \geqq u_c - u_a$ for $\hat{r} \neq 0$ we obtain the differential inequality

$$\frac{1}{2} \frac{\partial}{\partial t}(\|\hat{u}\|_\sigma^2) \leqq -\left\| \frac{\partial \hat{u}}{\partial z} \right\|_1^2 - \mu^2 \left\| \frac{\partial \hat{w}}{\partial z} \right\|_1^2 + \int_{-\infty}^{+\infty} \hat{r} \hat{u}^2 \left(\lambda_c - \frac{\lambda}{2} \right) dz.$$

Thus, for $\lambda \geqq 2\lambda_c$, we have

$$\frac{\partial}{\partial t}\left\{\int_{-\infty}^{+\infty} \sigma \hat{u}^2 \, dz\right\} \leqq 0.$$

Since σ is necessarily a positive function, we deduce that $\|\hat{u}\|_{\sigma}^2$ must tend to a limiting value as $t \to \infty$. Furthermore, unless $\hat{u}$ and $\hat{w}$ are both constant, strict inequality holds. Thus, since the only admissible constant value for $\hat{u}$ is zero, the result follows.

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