Persistence of Invariant Sets for Dissipative Evolution Equations

Don A. Jones[*]

Department of Mathematics, Arizona State University, Tempe, Arizona 85287-1804

Andrew M. Stuart[†]

SCCM Program, Division of Mechanics and Computation, Durand 257, Stanford University, Stanford, California 94305-4040

and

Edriss S. Titi[‡]

Department of Mathematics and Department of Aerospace Engineering, University of California, Irvine, California 92697-3875

We show that results concerning the persistence of invariant sets of ordinary differential equations under perturbation may be applied directly to a certain class of partial differential equations. Our framework is particularly well-suited to encompass numerical approximations of these partial differential equations. Specifically, we show that for a class of PDEs with a C^1 inertial form, certain natural numerical approximations possess an inertial form close to that of the underlying PDE in the C^1 norm. © 1998 Academic Press

1. INTRODUCTION

We consider a class of evolution partial differential equations (PDEs) that are known to possess an inertial manifold. An inertial manifold is

*E-mail: dajones@math.la.asu.edu.
†E-mail: stuart@sccm.stanford.edu.
‡E-mail: etiti@math.uci.edu.

479

a *smooth* finite-dimensional, exponentially attracting, positively invariant manifold (which contains the global attractor), first introduced in Foias, Sell, and Temam [14, 15]. See also Constantin, Foias, Nicolaenko, and Temam [6, 7], Fabes, Luskin, and Sell [11], Foias *et al.* [13], Foias, Sell, and Titi [16], Mallet-Paret and Sell [30], and Sell and You [35]. Thus, while the dynamical system generated by the solutions of the PDE is infinite dimensional, the reduction of the PDE to its inertial manifold yields a finite-dimensional system, called an inertial form, whose long-time behavior is the same as that of the PDE. In particular, this finite-dimensional system has the same bounded invariant sets as the original PDE. Thus one might expect that results concerning the persistence of invariant sets under perturbation derived in the context of ordinary differential equations (finite-dimensional systems) might apply directly to infinite-dimensional systems that possess inertial manifolds. Furthermore, of particular interest in the case of PDEs is to understand the behavior of bounded invariant sets under numerical approximation.

In general, we consider PDEs that may be expressed as an evolution equation,

$$\frac{du}{dt} + Au + R(u) = 0, \tag{1.1}$$

$$u(0) = u_0$$

on a Hilbert space H. We denote the inner product in H by $(\cdot, \cdot)$ and norm $|\cdot|^2 = (\cdot, \cdot)$. We assume that A is a densely defined sectorial linear operator with compact inverse. Thus it is possible to choose $\zeta \geq 0$ such that all eigenvalues of $\tilde{A} := A + \zeta I$ have strictly positive real part. For $0 < \alpha < 1$ we define $\tilde{A}^\alpha = (\tilde{A}^{-\alpha})^{-1}$, where

$$\tilde{A}^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\tilde{A}t} dt. \tag{1.2}$$

We denote by $D(\tilde{A}^\alpha)$ the domain of $\tilde{A}^\alpha$. See Henry [19] and Pazy [32]. For $\alpha = 0$ we define $\tilde{A}^0 = I$. Then $D(\tilde{A}^\alpha)$ is a Hilbert space with the inner product $(\tilde{A}^\alpha u, \tilde{A}^\alpha v)$ and norm $|u|_\alpha = |\tilde{A}^\alpha u|$ for all $u, v \in D(\tilde{A}^\alpha)$.

The operator A generates an analytic semigroup $L(t)$. We assume that $R(u)$ satisfies sufficient conditions so that the initial value problem (1.1) generates a semigroup $S(t) : D(\tilde{A}^\gamma) \to D(\tilde{A}^\gamma)$ for some $\gamma \geq 0$, which is locally Lipschitz continuous on $D(\tilde{A}^\gamma)$ for each $t \geq 0$. That is, $|S(t)u - S(t)v|_\gamma \leq C|u - v|_\gamma$, where C depends on the norms of $u, v \in D(\tilde{A}^\gamma)$ and t. We suppose further that the system (1.1) is dissipative. That is there exists a ball of radius R in $D(\tilde{A}^\gamma)$, $B(0, R)$, that is absorbing: for every $r > 0$ there exists a $T(r) \geq 0$ such that $S(t)B(0, r) \subset B(0, R)$ for all $t \geq T$ (see Hale [18] and Temam [37]).

We employ the notation

$$\|u\| := |u|_\gamma \qquad \forall u \in D(\tilde{A}^\gamma)$$

and

$$\|T\|_{op} := \sup_{\substack{u \in D(\tilde{A}^\gamma) \\ \|u\|=1}} |Tu|_\gamma$$

for a linear operator $T \in \mathcal{L}(D(\tilde{A}^\gamma))$ and when γ is fixed in an argument. Let Λ be any real number. We denote by H_Λ the finite-dimensional space spanned by all of the generalized eigenfunctions of the operator A with corresponding eigenvalues with real part less than or equal to Λ. We denote by P_Λ the projection of H onto H_Λ and by $Q_\Lambda = I - P_\Lambda$.

Let $\tau > 0$. The solution of (1.1) $u(\tau) = S(\tau)u_0$ may be written as a map

$$G(\tau, u) := L(\tau)u + N(\tau, u), \tag{1.3}$$

where

$$L(\tau)u := e^{-A\tau}u, \qquad N(\tau, u) := -\int_0^\tau L(\tau - s)R(S(s)u)ds. \tag{1.4}$$

For convenience we frequently drop the explicit dependence of the map G on τ. If M is an invariant bounded subset of the solution operator $S(t)$, it is invariant under G as well. On the other hand for any bounded invariant subset M of G, the set $\tilde{M} = \bigcup_{0 \le t \le \tau} S(t)M$ is a bounded invariant subset of $S(t)$. It therefore suffices to study the behavior of the invariant sets of G under perturbation.

In this paper we apply results concerning the persistence of bounded invariant sets derived for finite-dimensional systems to the map G, which could be defined on an infinite-dimensional space. To accomplish this we will seek a finite-dimensional system that has the same bounded invariant sets as does (1.3). In particular we will require the map G to have an inertial manifold. To date inertial manifolds have been constructed as the graph of a smooth function, $\Phi : P_\Lambda H \to Q_\Lambda H$ for some suitably chosen Λ (Φ is shown to be C^1 in Chow, Lu, and Sell [5], Demengel and Ghidaglia [9], Ou and Sritharan [31], and Sell and You [35] under essentially the same conditions as for the existence of an inertial manifold). If the map G has an inertial manifold of this type, the reduction of the map G to this manifold provides a map defined on the finite-dimensional space $P_\Lambda H$, given by

$$p_{n+1} = Lp_n + P_\Lambda N(p_n + \Phi(p_n)) = P_\Lambda G(p_n + \Phi(p_n)). \tag{1.5}$$

This map is called an inertial form, and it has the same dynamics as G on the inertial manifold.

The map G_h might originate from a fully discrete or semi discrete approximation to (1.1). Our main assumptions on the map G_h are that G and G_h are close in the C^1 norm on bounded sets that contain all of the bounded invariant sets of the map G. We emphasize that if the map G_h is a numerical approximation of the PDE, our assumptions on G_h are extensions to a C^1 setting of classical estimates indicating that trajectories of G_h approximate trajectories of the PDE over finite-time intervals.

The task is to write the map G_h in the form of (1.5), and in such a way that it is close to (1.5) in the C^1 norm. Since we compare maps defined on finite-dimensional spaces, results concerning the persistence of invariant sets studied in the context of finite-dimensional systems under perturbation may be obtained and then extended directly to the map G, and hence, to the PDE and its numerical approximation. These results include Beyn [3], Beyn and Lorenz [4], Fenichel [12], Hirsh, Pugh, and Shub [21], Kloeden and Lorenz [26], Pliss and Sell [33], Sacker [34], and Stuart [36]. Some of these have already been extended to infinite-dimensional systems. In particular, the work of Fenichel in Jones and Shkoller [23] and the work of Hirsh, Pugh, and Shub in Bates, Lu, and Zeng [2].

The technical results in this work show that the inertial manifold persists under C^1 perturbation. We are, however, mainly interested in the consequences of this persistence. The essential improvements on previous studies are the generality of the setting and the extension to C^1 approximation. In many cases numerical schemes approximating PDEs that have inertial manifolds are already known to have inertial manifolds. This was first shown in Foias, Sell, and Temam [15] and in Foias, Sell, and Titi [16] for a Galerkin approximation of the underlying PDE. Demengel and Ghidaglia [10] looked at the behavior of the inertial manifold under a time discretization. Foias and Titi [17] studied a finite difference approximation. Jones and Stuart [24] studied maps of the form (1.3): under appropriate conditions, the map is shown to possess an inertial manifold. Approximations of this map, which are also studied here, are shown to possess inertial manifolds in that work. The approximations of G we consider may be interpreted as being a fully discrete approximation of the PDE generating the map G. The approximations may include finite difference, finite element, spectral approximations and their time discretizations. The results and conclusions of this work have been previously presented for some specific PDEs and numerical algorithms. See Jones [22] and Jones and Titi [25].

Many results concerning the relationship between the long-time behavior of numerical schemes and the PDE they approximate already exist. They include Constantin, Foias, and Temam [8] and Titi [38] where sufficient conditions on the Galerkin scheme are derived to infer the existence of a nearby stable stationary solution for the Navier–Stokes equations from the apparent stability of the time-dependent Galerkin approximate solutions;

stable periodic orbits are studied analogously in Titi [39]. The relation of the large-time behavior of finite element approximations to the solutions of the Navier–Stokes equations is considered in Heywood and Rannacher [20]. Other works followed that studied the behavior of solutions near equilibrium of PDEs under numerical approximation. They include Alouges and Debussche [1], Larsson and Sanz-Serna [27, 28], and Stuart [36]. We expect that many of these results will hold for PDEs with inertial manifolds as a consequence of our main theorem.

We now outline the paper. In Section 2 we give sufficient conditions for a map G to have an inertial manifold representable as the graph of a C^1 function. In Section 3 we study the behavior of the manifold under C^1 perturbation. The consequences for the persistence of invariant sets are discussed in Section 4. We conclude in Section 5 with an application to a PDE, a Galerkin spatial discretization, and a semi-implicit Euler time discretization.

2. THE MAP G

We consider a general map G that depends on a parameter $\tau > 0$, of the form

$$G(\tau, u) = L(\tau)u + N(\tau, u). \tag{2.1}$$

Such a map was motivated in the previous section, but it need not be (1.3). Again we may drop the explicit dependence of the map on τ when no confusion may arise. We suppose further that the map G has the following properties.

2.1. *Assumptions* (G).

The map G is dissipative with the ball $B(0, R) \subset D(\tilde{A}^\gamma)$ absorbing for some $R > 0$. The linear operator $L \in \mathscr{L}(D(\tilde{A}^\gamma))$ and for every real Λ the spaces $P_\Lambda H$ and $Q_\Lambda H$ are invariant subspaces for L. Moreover, there exist positive constants a, b depending on Λ, τ, positive constants K_1, K_2 depending on Λ, τ, R, and $c \in (0, 1)$ depending on τ only such that

$$b\|p\| \leq \|Lp\| \leq c\|p\| \qquad \forall p \in P_\Lambda H, \tag{G1}$$

$$\|Lq\| \leq a\|q\| \qquad \forall q \in Q_\Lambda H. \tag{G2}$$

We assume that $N \in C^1(D(\tilde{A}^\gamma), D(\tilde{A}^\gamma))$. More specifically, we assume that

$$\|\mathscr{R}N(u)\| \leq K_1, \qquad \|\mathscr{R}DN(u)w\| \leq K_2\|w\| \quad \forall u, w \in D(\tilde{A}^\gamma), \tag{G3}$$

where $\mathscr{R} = I$, P_Λ, or Q_Λ, and DN is the Fréchet derivative of N. Moreover, there is an $R_N > R$ such that $N(u) = 0$, $DN(u) = 0$ for all $u \in D(\tilde{A}^\gamma)$: $\|u\| \geq R_N$. Finally, we assume that $\|\mathscr{R}\| \leq \sigma(\Lambda)$, and where $\sigma(\Lambda)$ is a positive constant depending on Λ.

2.2. *Conditions* (C).

Let $\epsilon, \ell \in (0, \infty)$, and $\mu \in (0, 1)$, be arbitrarily chosen but fixed. In addition, let $B_i = 2\sigma(\Lambda)K_i$, $i = 1, 2$ (the reason for enlarging the constants K_1, K_2 will become clear when we turn to perturbations of G in Section 3). We assume that there exist real Λ and $\tau > 0$ depending on ϵ, ℓ such that the following inequalities hold,

$$4B_2(1 + \ell) \leq b - a, \tag{C1}$$

$$a\epsilon + B_1 \leq \epsilon, \tag{C2}$$

$$\theta := a\,\ell + B_2(1 + \ell) \leq \ell\phi, \tag{C3}$$

where $\phi := b - B_2(1 + \ell) > 0$ (notice that by (C1), $\phi > 0$), and

$$a + B_2(1 + \ell) \leq \mu. \tag{C4}$$

Condition (C1) is equivalent to the well known gap condition in the real part of the spectrum of the operator A which is sufficient for the existence of an inertial manifold (see the references above). The verification of (C2) corresponds to the operator A having eigenvalues with sufficiently large real part, while (C3) again corresponds to sufficiently large spectral gaps in the real part of the spectrum of A. The condition (C4) requires that the gap in the spectrum occurs for eigenvalues with sufficiently large real part.

We have from Jones and Stuart [24]

THEOREM 2.1. *Under assumptions* (G) *and Conditions* (C) *the map G has an inertial manifold representable as the graph of a function $\Phi : P_\Lambda H \to Q_\Lambda H$. Moreover, $\Phi(p) = 0$ for $p \in P_\Lambda H : \|p\| \geq cR_N + B_1$, and*

$$\sup_{p \in P_\Lambda H} \|\Phi(p)\| \leq \epsilon, \qquad \|\Phi(p_1) - \Phi(p_1)\| \leq \ell\|p_1 - p_2\| \ \forall p_1, p_2 \in P_\Lambda H.$$

Remark. In the case that the map G is (1.3), the inertial manifold given by Theorem 2.1 is an inertial manifold for the PDE (1.1). That is, the manifold is independent of the parameter τ.

The proof of Theorem 2.1 will essentially be repeated below when we show a stronger result, namely that the map Φ is continuously differentiable. The basic idea is to look for a fixed point of the map T_1 defined below on the space

$$\Gamma(\epsilon, \ell) = \Big\{ \Psi \in C(P_\Lambda H, Q_\Lambda H) : \sup_{p \in P_\Lambda H} \|\Psi(p)\| \leq \epsilon,$$

$$\Psi(p) = 0, \ \forall p \in P_\Lambda H : \|p\| \geq cR_N + B_1,$$

$$\|\Psi(p_1) - \Psi(p_2)\| \leq \ell\|p_1 - p_2\| \ \forall p_1, p_2 \in P_\Lambda H \Big\}.$$

Let $\Psi \in \Gamma(\epsilon, \ell)$. Then for any $p \in P_\Lambda H$ one first shows that there exists a unique $\xi(p, \Psi) \in P_\Lambda H$ such that

$$p = L\xi + P_\Lambda N(\xi + \Psi(\xi)) \tag{2.2}$$

holds. The map T_1 is then defined to be

$$(T_1\Psi)(p) = L\Psi(\xi) + Q_\Lambda N(\xi + \Psi(\xi)).$$

That such a ξ exists solving (2.2) is shown in Jones and Stuart [24] and follows from Assumptions (G) and Conditions (C) (see Lemma 2.3 below for a related argument). Moreover, the map T_1 is shown to be a contraction on $\Gamma(\epsilon, \ell)$. In particular, we have for μ given in Conditions (C), $\Psi_1, \Psi_2 \in \Gamma(\epsilon, \ell)$ that

$$\sup_{p \in P_\Lambda H} \|T_1\Psi_1(p) - T_1\Psi_2(p)\| \le \mu \sup_{p \in P_\Lambda H} \|\Psi_1(p) - \Psi_2(p)\|. \tag{2.3}$$

To show $\Phi \in C^1$ we use a standard technique (see, for example, Fenichel [12], Hirsh, Pugh and Shub [21], and Chow, Lu, and Sell [5]): we formulate a map for which $D\Phi$ would be the fixed point if Φ were differentiable. Using a contraction argument we then show that such a map indeed has a fixed point which is the Fréchet derivative of Φ, $D\Phi$. To better motivate this map we first prove

LEMMA 2.2. *Suppose that Conditions* (C) *and Assumptions* (G) *hold. Suppose further that the fixed point of the map T_1, Φ, is differentiable and that $\sup_{p \in P_\Lambda H} \|D\Phi(p)\|_{op} \le \ell$. Then $\xi(p)$ solving* (2.2) *is differentiable and $\eta = \frac{D\xi}{Dp}\rho$, $\rho \in P_\Lambda H$ satisfies*

$$\rho = L\eta + P_\Lambda DN(\xi + \Phi(\xi))(I + D\Phi(\xi))\eta$$

$$D\Phi(p)\rho = LD\Phi(\xi)\eta + Q_\Lambda DN(\xi + \Phi(\xi))(I + D\Phi(\xi))\eta.$$

Proof. Consider the function $F(p, \xi) = p - L\xi - P_\Lambda N(\xi + \Phi(\xi))$. According to (2.2) we know that there exists a unique $\xi \in P_\Lambda H$ such that $F(p, \xi(p)) = 0$. We would like to implement the implicit function theorem to show that $D\xi/Dp$ exists. In fact all we need to show is that $DF/D\xi$ is invertible in a neighborhood of $(p, \xi(p))$ for all $p \in P_\Lambda H$, since $N(\cdot + \Phi(\cdot))$ is C^1. However, we are able to show that $DF/D\xi$ is invertible on all of $P_\Lambda \times P_\Lambda$. Since L in invertible on $P_\Lambda H$, we need only show that $L^{-1}P_\Lambda DN(\xi + \Phi(\xi))(I + D\Phi(\xi))$ has norm less than one for all $\xi \in P_\Lambda H$. Using (G1) and (C1) we find that

$$\|L^{-1}P_\Lambda DN(\xi + \Phi(\xi))(I + D\Phi(\xi))\eta\| \le B_2(1 + \ell)\|\eta\| \le \frac{1}{4}\|\eta\|.$$

Now that we have established that ξ is differentiable, we differentiate (2.2) to obtain the system given in the lemma. ∎

Consider the complete metric space

$$\Gamma(\ell) := \Big\{ \Upsilon \in C(P_\Lambda H, L(P_\Lambda H, Q_\Lambda H)) : D\Upsilon(p) = 0,$$

$$\forall p \in P_\Lambda H : \|p\| \geq cR_N + B_1, \ \|\Upsilon\|_\Gamma := \sup_{p \in P_\Lambda H} \|\Upsilon(p)\|_{op} \leq \ell \Big\}.$$

Motivated by the above lemma we define a map T_Ψ on $\Gamma(\ell)$ as follows. Given any fixed $\Psi \in \Gamma(\epsilon, \ell)$ and any $p, \rho \in P_\Lambda H$ we first find a unique $\eta(p, \Psi, \Upsilon) \in P_\Lambda H$ such that for $\xi \in P_\Lambda H$ solving (2.2)

$$\rho = L\eta + P_\Lambda DN(\xi + \Psi(\xi))(I + \Upsilon(\xi))\eta. \tag{2.4}$$

The map T_Ψ is then given by

$$(T_\Psi \Upsilon(p))\rho = L\Upsilon(\xi)\eta + Q_\Lambda DN(\xi + \Psi(\xi))(I + \Upsilon(\xi))\eta. \tag{2.5}$$

That for any $\rho \in P_\Lambda H$ there is a unique $\eta \in P_\Lambda H$ solving (2.4) is established by

LEMMA 2.3. *Suppose that Conditions* (C) *and Assumptions* (G) *hold. Given* $\rho \in P_\Lambda H$, $\Psi \in \Gamma(\epsilon, \ell)$, *and* $\Upsilon \in \Gamma(\ell)$, *there exists a unique* $\eta \in P_\Lambda H$ *such that* (2.4) *holds. Moreover, we have that*

$$\|\eta\| \leq \frac{\|\rho\|}{\phi}, \tag{2.6}$$

where ϕ *is given after* (C3).

Proof. Since (2.4) is linear and L has a bounded inverse on $P_\Lambda H$ (by Assumptions (G)), we need only show that $L^{-1}P_\Lambda DN(\xi + \Psi(\xi))(I + \Upsilon(\xi))$ has norm less than one. This estimate is given in Lemma 2.2.

Directly from (2.4) and (G1) we obtain that

$$\|\eta\| \leq b^{-1}\|\rho - P_\Lambda DN(\xi + \Psi(\xi))(I + \Upsilon(\xi))\eta\|$$

$$\leq b^{-1}\|\rho\| + b^{-1}B_2(1 + \ell)\|\eta\|.$$

Using (C3) we find that (2.6) holds. Thus the map T_Ψ defined by (2.5) is well-defined. ∎

LEMMA 2.4. *Suppose that Conditions* (C) *and Assumptions* (G) *hold. Then the map* (2.5) *maps* $\Gamma(\ell)$ *into* $\Gamma(\ell)$. *Moreover,* T_Ψ *is a contraction on* $\Gamma(\ell)$ *for each* $\Psi \in \Gamma(\epsilon, \ell)$ *with unique fixed point denoted by* Υ_Ψ, *and this fixed point is continuous with respect to* Ψ.

Proof. Let $\Upsilon \in \Gamma(\ell)$. Then using (G2), (G3), (C3), and (2.6) we have that

$$\|T_\Psi \Upsilon(p)\rho\| \leq a\,\ell\|\eta\| + B_2(1+\ell)\|\eta\|$$
$$\leq [a\,\ell + B_2(1+\ell)]\frac{\|\rho\|}{\phi} \leq \ell\|\rho\|. \tag{2.7}$$

Hence, $\|T_\Psi \Upsilon\|_\Gamma \leq \ell$.

To show that $T_\Psi \Upsilon(p) = 0$ for all $p \in P_\Lambda H : \|p\| \geq cR_N + B_1$ notice that for such p we have, using (2.2) and Assumptions (G), that

$$cR_N + B_1 \leq \|p\| \leq c\|\xi\| + B_1. \tag{2.8}$$

Hence, $R_N \leq \|\xi\|$ and $N(\xi + \Psi(\xi)) = 0$, $DN(\xi + \Psi(\xi)) = 0$. Repeating the calculation leading to (2.8), we find, using Assumptions (G), that $cR_N + B_1 \leq c\|\xi\| \leq \|\xi\|$. By assumption $\Upsilon(\xi) = 0$ for such ξ, and hence so does $T_\Psi(p)$ for all $\|p\| \geq cR_N + B_1$.

We need to show that $T_\Psi \Upsilon$ is continuous. Let $p_0 \in P_\Lambda H$ be arbitrarily chosen and fixed. We need to show that $\|T_\Psi \Upsilon(p_0) - T_\Psi \Upsilon(p_1)\|_{op}$ may be made arbitrarily small by requiring $\|p_0 - p_1\|$ sufficiently small. By Lemma 2.3 and Lemma 2.3 in Jones and Stuart [24], we may chose $\xi_i, \eta_i \in P_\Lambda H$, $i = 0, 1$ so that

$$p_i = L\xi_i + P_\Lambda N(\xi_i + \Psi(\xi_i)) \tag{2.9}$$

$$\rho = L\eta_i + P_\Lambda DN(\xi_i + \Psi(\xi_i))(I + \Upsilon(\xi_i))\eta_i \tag{2.10}$$

$$(T_\Psi \Upsilon(p_i))\rho = L\Upsilon(\xi_i)\eta_i + Q_\Lambda DN(\xi_i + \Psi(\xi_i))(I + \Upsilon(\xi_i))\eta_i, \tag{2.11}$$

for $i = 0, 1$ and for arbitrary $\rho \in P_\Lambda H$ such that $\|\rho\| = 1$. We find, using (2.9), (G1), (G3), and the properties of $\Psi \in \Gamma(\epsilon, \ell)$, that

$$\|\xi_1 - \xi_0\| \leq b^{-1}\big[\|p_1 - p_0\| + B_2(1+\ell)\|\xi_1 - \xi_0\|\big].$$

From (C3) we find

$$\|\xi_1 - \xi_0\| \leq \frac{\|p_1 - p_0\|}{\phi}. \tag{2.12}$$

A similar calculation using (2.10) shows that

$$\|\eta_1 - \eta_0\| \leq \frac{1}{\phi^2}\Big(B_2\|\Upsilon(\xi_1) - \Upsilon(\xi_0)\|_{op}$$
$$+ (1+\ell)\|P_\Lambda DN(\xi_1 + \Psi(\xi_1)) - P_\Lambda DN(\xi_0 + \Psi(\xi_0))\|_{op}\Big),$$

where we have also used (2.6) to obtain $\|\eta_i\| \leq \|\rho\|\phi^{-1} \leq \phi^{-1}$ for $i = 0, 1$. Since Ψ, Υ and DN are continuous by assumption, both $\|\xi_1 - \xi_0\|$, $\|\eta_1 - \eta_0\|$ can be made arbitrarily small independently of ρ by requiring $\|p_1 - p_0\|$ to be sufficiently small. Using this fact one may obtain directly from (2.11) that $T_\Psi \Upsilon(\cdot)$ is continuous.

We show now that the map T_Ψ is a contraction on $\Gamma(\ell)$ uniformly in $\Psi \in \Gamma(\epsilon, \ell)$. That is, the Lipschitz constant is less than one, and it is independent of the specific choice of Ψ. Let $p, \rho \in P_\Lambda H$, with $\|\rho\| = 1$, ξ solve (2.2). Then by Lemma 2.3 there exists η_i $i = 1, 2$ such that

$$\rho = L\eta_i + P_\Lambda DN(\xi + \Psi(\xi))(I + \Upsilon_i(\xi))\eta_i \qquad (2.13)$$

$$(T_\Psi \Upsilon_i(p))\rho = L\Upsilon_i(\xi)\eta_i + Q_\Lambda DN(\xi + \Psi(\xi))(I + \Upsilon_i(\xi))\eta_i.$$

We obtain directly from (2.13) that

$$\begin{aligned} -L(\eta_1 - \eta_2) = {} & P_\Lambda DN(\xi + \Psi(\xi))(\Upsilon_1(p) - \Upsilon_2(p))\eta_1 \\ & + P_\Lambda DN(\xi + \Psi(\xi))(I + \Upsilon_2(p))(\eta_1 - \eta_2). \end{aligned}$$

Estimating as above, using Assumptions (G), Conditions (C) and (2.6), we find that

$$\|\eta_1 - \eta_2\| \leq \frac{B_2 \|\rho\|}{\phi^2} \|\Upsilon_1 - \Upsilon_2\|_\Gamma. \qquad (2.14)$$

We have that

$$\begin{aligned} (T_\Psi \Upsilon_1(p) - T_\Psi \Upsilon_2(p))\rho = {} & L(\Upsilon_1(p) - \Upsilon_2(p))\eta_1 + L\Upsilon_2(p)(\eta_1 - \eta_1) \\ & + Q_\Lambda DN(\xi + \Psi(\xi))[\Upsilon_1(p) - \Upsilon_2(p)]\eta_1 \\ & + Q_\Lambda DN(\xi + \Psi(\xi))[I + \Upsilon_2(p)](\eta_1 - \eta_1). \end{aligned}$$

Estimating as before we find that

$$\|(T_\Psi \Upsilon_1(p) - T_\Psi \Upsilon_2(p))\rho\| \leq \frac{a + B_2(1 + \ell)}{\phi} \|(\Upsilon_1 - \Upsilon_2)\|_\Gamma.$$

However, from (C1), $a + B_2(1 + \ell) \leq a + (b - a)/4$ and $\phi = b - B_2(1 + \ell) \geq b - (b - a)/4$. Thus

$$\begin{aligned} \|(T_\Psi \Upsilon_1(p) - T_\Psi \Upsilon_2(p))\rho\| & \leq \frac{a + (1/4)(b - a)}{b - (1/4)(b - a)} \|(\Upsilon_1 - \Upsilon_2)\|_\Gamma \\ & := \mu_c \|(\Upsilon_1 - \Upsilon_2)\|_\Gamma, \qquad (2.15) \end{aligned}$$

and $\mu_c < 1$. Hence, T_Ψ is a contraction on $\Gamma(\ell)$, and there is a unique $\Upsilon \in \Gamma(\ell)$ such that $T_\Psi \Upsilon = \Upsilon$.

Finally, we show that the unique fixed point of T_Ψ denoted Υ_Ψ is continuous in Ψ. We obtain from (2.15) that

$$\|\Upsilon_{\Psi_1} - \Upsilon_{\Psi_2}\|_\Gamma \leq \frac{1}{1-\mu_c}\|T_{\Psi_1}\Upsilon_{\Psi_1} - T_{\Psi_2}\Upsilon_{\Psi_1}\|_\Gamma.$$

To show that the right hand of this last inequality tends to zero as $\sup_{p \in P_\Lambda H}\|\Psi_1 - \Psi_2\| \to 0$ is similar to the proof that $T_\Psi(p)$ in continuous in p shown above. We omit the details. However, the estimates require that N, DN, Ψ, Υ are all uniformly continuous. Since they all have compact support, this readily follows. ∎

THEOREM 2.5. *Under Assumptions* (G) *and Conditions* (C) *the inertial manifold for the map G is C^1.*

Proof. We need only to show that Φ given in Theorem 2.1 is C^1. Let Υ_Φ be the fixed point of T_Ψ with $\Psi = \Phi$, where the graph of Φ is the inertial manifold for the map G given in Theorem 2.1. We will show that Φ is differentiable and that $\Upsilon_\Phi = D\Phi$ is the Fréchet derivative of Φ. Suppose that $\Phi_0(p) \in \Gamma(\epsilon, \ell)$, $\Phi_0(p) \in C^1(P_\Lambda H, Q_\Lambda H)$ and that $\|D\Phi_0\|_\Gamma \leq \ell$. That is, $D\Phi_0 \in \Gamma(\ell)$. For example, we could take $\Phi_0 \equiv 0$. Set $\Upsilon_0 = D\Phi_0$. Let $\xi_n \in P_\Lambda H$ solve (2.2) with $\Psi = \Phi_n$ and $\eta_n \in P_\Lambda H$ solve (2.4) with $\Upsilon = D\Phi_n$, $\Psi = \Phi_n$ and where the sequence $\{\Phi_n\}$, $n \geq 0$, is defined iteratively by

$$\Phi_{n+1}(p) = L\Phi_n(\xi_n) + Q_\Lambda N(\xi_n + \Phi_n(\xi_n)). \tag{2.16}$$

As in Lemma 2.2 we may differentiate Φ_n to obtain

$$D\Phi_{n+1}(p)\rho = LD\Phi_n(\xi_n)\eta_n + Q_\Lambda DN(\xi_n + \Phi_n(\xi_n))(I + D\Phi_n(\xi_n))\eta_n.$$

Notice that $T_{\Phi_n}D\Phi_n = D\Phi_{n+1}$. Hence, $D\Phi_{n+1}$ is in $\Gamma(\ell)$ for $n \geq 0$.

We will show that $\{\Phi_n\}$ is a Cauchy sequence in $C^1(P_\Lambda H, Q_\Lambda H)$. Since $\{\Phi_n\}$ is Cauchy in $\Gamma(\epsilon, \ell)$ (which follows from (2.3)), it converges to Φ, the function whose graph is an inertial manifold for the map G in $\Gamma(\epsilon, \ell)$. Thus it suffices to show that $D\Phi_n$ is Cauchy in $\Gamma(\ell)$. The argument we use is similar to the one given in Lemma 4.1 in Chow, Lu, and Sell [5] and Hirsh, Pugh, and Shub [21]. Let Υ_{Φ_n} the fixed point of T_Ψ with $\Psi = \Phi_n$. Define $e_N = \sup_{m,n \geq N}\|\Upsilon_{\Phi_m} - \Upsilon_{\Phi_n}\|_\Gamma$. Thanks to Lemma 2.4 we have that $e_N \to 0$ as $N \to \infty$. Also we have for $n \geq N \geq 0$

$$\|D\Phi_{n+1} - \Upsilon_{\Phi_{n+1}}\|_\Gamma \leq \|D\Phi_{n+1} - \Upsilon_{\Phi_n}\|_\Gamma + \|\Upsilon_{\Phi_n} - \Upsilon_{\Phi_{n+1}}\|_\Gamma$$

$$\leq \|T_{\Phi_n}D\Phi_n - T_{\Phi_n}\Upsilon_{\Phi_n}\|_\Gamma + e_N$$

$$\leq \mu_c\|D\Phi_n - \Upsilon_{\Phi_n}\|_\Gamma + e_N.$$

An induction argument shows that for $n \geq m \geq N$

$$\|D\Phi_n - \Upsilon_{\Phi_n}\|_\Gamma \leq \mu_c^{m-N}\|D\Phi_N - \Upsilon_{\Phi_N}\|_\Gamma + \frac{e_N}{1-\mu_c}. \tag{2.17}$$

Therefore, we find that for such n, m

$$\|D\Phi_n - D\Phi_m\|_\Gamma \leq \|D\Phi_n - \Upsilon_{\Phi_n}\|_\Gamma + \|\Upsilon_{\Phi_n} - \Upsilon_{\Phi_m}\|_\Gamma + \|\Upsilon_{\Phi_m} - D\Phi_m\|_\Gamma$$

$$\leq 2\mu_c^{m-N}\|D\Phi_N - \Upsilon_{\Phi_N}\|_\Gamma + 2\frac{e_N}{1 - \mu_c} + e_N.$$

Then, for any $\tilde{\epsilon} > 0$, we may choose N large enough so that each of the last two terms on the right hand side of the last inequality are less than $\tilde{\epsilon}/3$. We then choose $m > N$ large enough so that the first term on the right hand side is less than $\tilde{\epsilon}/3$. This shows that the sequence $D\Phi_n$ is C^1-Cauchy. We conclude that $\Phi_n \to \Phi$ in $\Gamma(\epsilon, \ell)$ and $D\Phi_n \to D\Phi$ in $\Gamma(\ell)$ as $n \to \infty$. We conclude from (2.17) by passing to the limit that $D\Phi = \Upsilon_\Phi$. ∎

3. PERTURBATIONS OF G

In this section we study the behavior of bounded invariant sets of the map G under perturbation. Keeping in mind that we are mainly interested in numerical approximations of the PDE, we consider a map $G_{h,o}(u)$ which may be defined on a finite-dimensional subspace, X_1^h, of $D(\tilde{A}^\gamma)$. The space X_1^h could be spanned by polynomials as in a finite-element space approximation. Let P^h be the projection $P^h : H \to X_1^h$. The map $G_{h,o}(\tau, u)$ approximates the map G. For example, if G is the time τ map of the semigroup of (1.1), then $G_{h,o}(\tau, u)$ may be a fully discrete approximation to (1.1). We will find it convenient to express the map $G_{h,o}(\tau, u^h)$ as a map on $D(\tilde{A}^\gamma)$. We therefore define

$$G_h(\tau, v) := G_{h,o}(\tau, P^h v).$$

Then $G_h : D(\tilde{A}^\gamma) \to D(\tilde{A}^\gamma)$. Rather than making specific assumptions about the space X_1^h and the projection P^h, we make the following assumptions about the map G_h. We recall that $B(0, R) := \{u \in D(\tilde{A}^\gamma) : \|u\| \leq R\}$. We set $E(v) := G(\tau, v) - G_h(\tau, v)$ for $v \in D(\tilde{A}^\gamma)$. Dropping the explicit dependence on the parameter τ, we assume the following.

3.1. *Assumptions* (G^h).

- $\|G(v) - G_h(v)\| = \|E(v)\| \leq K(R)h \; \forall v \in B(0, 2R)$,
- $\|DG(v) - DG_h(v)\|_{op} = \|DE(v)\|_{op} \leq K(R)h \;\; \forall v \in B(0, 2R)$,
where DE is the Fréchet derivative of E,

where R is given in Assumptions (G). The Assumptions (G^h) require only that the map G_h approximate the map G in the C^1 norm over the set $B(0, 2R)$, which contains all of the bounded invariant sets of the map G.

While the map G is assumed to be dissipative, we cannot expect in general that the map G_h satisfying Assumptions (G^h) will be dissipative. In particular, the inertial manifold may not survive this perturbation. If the map G_h is dissipative, then Assumptions (G^h) are enough to guarantee that G_h has an inertial manifold. This follows from Theorem 3.1 below. See also [15, 16, 10, 17, 22, 24, 23, 25, 36]. Our goal, however, is to show that those bounded invariant sets known to persist under C^1 perturbation proven in the context of finite-dimensional dynamics persist for the map G and its perturbation G_h. To accomplish this we construct a system similar to the inertial form, (1.5), for the map G_h which agrees with G_h in a region containing the bounded invariant sets of G. This new map will be C^1 close to the inertial form for G, (1.5), over this region.

Let $\theta : \mathbb{R}^+ \to [0, 1]$ be a fixed C^1 function such that $\theta(x) = 1$ for $0 \le x \le 2$, $\theta(x) = 0$ for $x \ge 4$, and $|\theta'(x)| \le 2$ for $x \ge 0$. Define $\theta_R(x) = \theta(x/R^2)$. We consider the map

$$
\begin{aligned}
u = \tilde{G}_h(v) &:= Lv + N(v) - \theta_R(\|v\|^2)E(v) \\
&:= Lv + N^h(v),
\end{aligned}
\tag{3.1}
$$

where L is given in Assumptions (G). The map $\tilde{G}_h$ agrees with the map G_h inside the ball $B(0, R)$ and

$$
\begin{aligned}
\|G(v) - \tilde{G}_h(v)\| &\le K(R)h, \\
\|DG(v) - D\tilde{G}_h(v)\|_{op} &\le K(R)h \ \forall v \in D(\tilde{A}^\gamma).
\end{aligned}
\tag{3.2}
$$

That the map $\tilde{G}_h$ has an inertial manifold is a simple consequence of the existence proof of the inertial manifold for maps, Theorem 2.1. Indeed, we have

THEOREM 3.1.   *Under Assumptions* (G), (G^h), *and Conditions* (C), *there exists an $h_1 > 0$ depending on $\tau, \Lambda, \epsilon, \ell$ such that the map $\tilde{G}_h$ has an inertial manifold representable as a graph of a C^1 function $\Phi^h : P_\Lambda H \to Q_\Lambda H$. Moreover, $\Phi^h \in \Gamma(\epsilon, \ell)$, $D\Phi^h \in \mathbf{\Gamma}(\ell)$.*

Proof.   We want to show that $N^h(v) := N(v) - \theta_R(\|v\|^2)E(v)$ has the same properties as $N(v)$ given in Assumptions (G). Thanks to the properties of the nonlinear term $N(v)$ given in Assumptions (G), Assumptions (G^h), and the properties of the function $\theta(x)$, we have that

$$
\begin{aligned}
\|\mathscr{R}N^h(v)\| &\le \|\mathscr{R}N(v)\| + \theta_R(\|v\|^2)\|\mathscr{R}E(v)\| \\
&\le \sigma(\Lambda)(K_1 + K(R)h) \qquad \forall v \in D(\tilde{A}^\gamma),
\end{aligned}
$$

where $\mathscr{R} = P_\Lambda$, Q_Λ, or I and by Assumptions (G), $\|\mathscr{R}\| \leq \sigma(\Lambda)$ for $\mathscr{R} = P_\Lambda$, Q_Λ, or I. Similarly, for all $v \in D(\tilde{A}^\gamma)$

$$
\begin{aligned}
\|\mathscr{R} D N^h(v)\|_{op} &\leq \|\mathscr{R} D[N(v) - \theta_R(\|v\|^2) E(v)]\|_{op} \\
&\leq \|\mathscr{R} D N(v) - \theta_R(\|v\|^2) D E(v)\|_{op} + \|D\theta_R(\|v\|^2) \cdot \mathscr{R} E(v)\|_{op} \\
&\leq \sigma(\Lambda) \left(K_2 + K(R) h + \frac{8}{R} K(R) h \right).
\end{aligned}
$$

Without loss of generality we may assume that $R \geq 1$. Thus for some $h_1 > 0$, we obtain

$$
\|N^h(v)\| \leq B_1, \qquad \|DN^h(v)\|_{op} \leq B_2 \ \forall v \in D(\tilde{A}^\gamma), \tag{3.3}
$$

for all $h \leq h_1$.

These estimates show the reason for deriving Theorem 2.1 with the constants B_1, B_2 enlarged in Assumptions (G). Theorem 2.1 and Theorem 2.5 apply and the map $\tilde{G}_h$ has a C^1 inertial manifold. ∎

The next result shows that the inertial manifolds converge in the C^1 norm as $h \to 0$. From now on Graph(Φ) and Graph(Φ^h) are the inertial manifolds for the maps G and $\tilde{G}_h$, respectively.

THEOREM 3.2. *Suppose that Assumptions* (G), (G^h), *and Conditions* (C) *hold and that* $h \leq h_1$ *so that the map* $\tilde{G}_h$ *has an inertial manifold as in Theorem 3.1. Then for any* $\epsilon' > 0$, *there exists an* $h_0(\epsilon') > 0$ *such that*

$$
\sup_{p \in P_\Lambda H} \|\Phi(p) - \Phi^h(p)\| \leq Kh
$$

$$
\sup_{p \in P_\Lambda H} \|D\Phi(p) - D\Phi^h(p)\|_{op} \leq \epsilon'
$$

for all $h \leq h_0$.

Proof. The proof uses the uniform contraction principle. Let $p \in P_\Lambda H$ be given. We denote by Φ^h the function whose graph is an inertial manifold of the map $\tilde{G}_h$. Set

$$
p = L\xi + P_\Lambda N(\xi + \Phi(\xi))
$$

$$
(T_1 \Phi)(p) = L\Phi(\xi) + Q_\Lambda N(\xi + \Phi(\xi)),
$$

for some $\xi \in P_\Lambda H$ as in (2.2). Notice that $(T_1 \Phi)(p) = Q_\Lambda G(\xi + \Phi(\xi))$. With the same ξ we set

$$
p^h := L\xi + P_\Lambda N^h(\xi + \Phi(\xi)).
$$

The map T_1^h is defined exactly as T_1 was in Section 2, only N is replaced by N^h defined in (3.1). Let Φ and Φ^h the fixed points of T_1 and T_1^h, respectively. We have by the uniform contraction theorem, that is, from (2.3),

$$\sup_{p \in P_\Lambda H} \|\Phi(p) - \Phi^h(p)\| \le \frac{1}{1 - \mu} \sup_{p \in P_\Lambda H} \|T_1 \Phi(p) - T_1^h \Phi(p)\|. \tag{3.4}$$

However,

$$\|T_1 \Phi(p) - T_1^h \Phi(p)\| \le \|T_1 \Phi(p) - T_1^h \Phi(p^h)\| + \|T_1^h \Phi(p^h) - T_1^h \Phi(p)\|.$$

We have that $\|T_1 \Phi(p) - T_1^h \Phi(p^h)\| = \|Q_\Lambda[G(\xi + \Phi(\xi)) - \tilde{G}_h(\xi + \Phi(\xi))]\|$, which by Assumptions (G), (G^h), and (3.2) is less than or equal to $\sigma(\Lambda) K(R) h$. The previous section shows that the map $T_1 \Phi(\cdot)$ is differentiable and its derivative is bounded (Lemma 2.4). Thus by the mean value theorem $\|T_1^h \Phi(p^h) - T_1^h \Phi(p)\| \le K \|p^h - p\|$. Notice that $\|p^h - p\|$ is $\|P_\Lambda[G(\xi + \Phi(\xi)) - \tilde{G}_h(\xi + \Phi(\xi))]\|$. Thus from Assumptions (G^h) this is less than or equal to $\sigma(\Lambda) K(R) h$ for all $p \in P_\Lambda H$. Returning to (3.4) the first inequality in the proposition follows.

The second inequality in the proposition follows in a similar manner. Let $\rho \in P_\Lambda H$ with $\|\rho\| = 1$ given. By Lemma 2.3 there exists $\eta \in P_\Lambda H$ such that

$$\rho = L\eta + P_\Lambda DN(\xi + \Phi(\xi))(I + D\Phi(\xi))\eta$$

$$D\Phi(p)\rho = L\Phi(\xi)\eta + Q_\Lambda DN(\xi + \Phi(\xi))(I + D\Phi(\xi))\eta,$$

where ξ is as in the first part of the proof. We have that

$$D\Phi(p)\rho = Q_\Lambda DG(\xi + \Phi(\xi))(I + D\Phi(\xi))\eta.$$

Set

$$\rho^h := L\eta + P_\Lambda DN^h(\xi + \Phi(\xi))(I + D\Phi(\xi))\eta.$$

The map T_Ψ^h is defined exactly as T_Ψ was in Section 2 only N is replaced with N^h defined by (3.1). The maps $D\Phi$ and $D\Phi^h$ are the fixed points of T_Φ and $T_{\Phi^h}^h$, respectively. We obtain by the uniform contraction theorem

$$\sup_{p \in P_\Lambda H} \|D\Phi(p) - D\Phi^h(p)\|_{op}$$

$$\le \frac{1}{1 - \mu_c} \sup_{p \in P_\Lambda H} \|T_\Phi D\Phi(p) - T_{\Phi^h}^h D\Phi(p)\|_{op}, \tag{3.5}$$

where μ_c is defined in (2.15).

We estimate the term on the right hand side as

$$T_\Phi D\Phi(p)\rho - T^h_{\Phi^h} D\Phi(p)\rho = \left(T_\Phi D\Phi(p)\rho - T^h_\Phi D\Phi(p^h)\rho^h\right)$$
$$+ \left(T^h_\Phi D\Phi(p^h)\rho^h - T^h_{\Phi^h} D\Phi(p)\rho^h\right)$$
$$+ T^h_{\Phi^h} D\Phi(p)(\rho^h - \rho)$$
$$:= E_1 + E_2 + E_3.$$

The first term E_1 is $Q_\Lambda(DG(\xi + \Phi(\xi)) - D\tilde{G}_h(\xi + \Phi(\xi)))(I + D\Phi(\xi))\eta$, which can be majorized using Assumptions (G^h), (3.2), (2.6). In particular, $\|E_1\| \le \sigma(\Lambda)K(R)(1 + \ell)\phi^{-1}h$ for all $p \in P_\Lambda H$, and all $\rho \in P_\Lambda H : \|\rho\| = 1$.
The two terms in E_2 are

$$T^h_\Phi D\Phi(p^h)\rho^h = LD\Phi(\xi)\eta + Q_\Lambda DN^h(\xi + \Phi(\xi))(I + D\Phi(\xi))\eta,$$

$$T^h_{\Phi^h} D\Phi(p)\rho^h = LD\Phi(\xi_1)\eta_1 + Q_\Lambda DN^h(\xi_1 + \Phi^h(\xi_1))(I + D\Phi(\xi_1))\eta_1.$$

In the first term

$$p^h = L\xi + P_\Lambda N^h(\xi + \Phi(\xi))$$

$$\rho^h = L\eta + P_\Lambda DN^h(\xi + \Phi(\xi))(I + D\Phi(\xi))\eta,$$

as above. In the second ξ_1, η_1 are such that (with p, ρ the same as above)

$$p = L\xi_1 + P_\Lambda N^h(\xi_1 + \Phi^h(\xi_1))$$

$$\rho^h = L\eta_1 + P_\Lambda DN^h(\xi_1 + \Phi^h(\xi_1))(I + D\Phi(\xi_1))\eta_1.$$

Just as in the derivation of (2.12) we have that $\|\xi_1 - \xi\| \le \phi^{-1}(\|p - p^h\| + \|\Phi(\xi_1) - \Phi^h(\xi_1)\|)$, which can be majorized by $\sigma(\Lambda)K(R)h$ for all $p \in P_\Lambda H$ by using Assumptions (G^h) and the first estimate in the lemma. Similarly, one finds by adding and subtracting appropriate terms that

$$\|\eta_1 - \eta\| \le \phi^{-1}\big\|P_\Lambda DN^h(\xi_1 + \Phi(\xi_1))(I + D\Phi(\xi_1))$$
$$- P_\Lambda DN^h(\xi + \Phi(\xi))(I + D\Phi(\xi))\big\|_{op}\|\eta\|$$
$$+ \phi^{-1}(1 + \ell)\big|\big\|P_\Lambda DN^h(\xi_1 + \Phi^h(\xi_1))$$
$$- P_\Lambda DN^h(\xi_1 + \Phi(\xi_1))\big\|_{op}\|\eta_1\|.$$

Recall that $N^h(v) = N(v) - \theta_R(\|v\|^2)E(v)$. Moreover, $N(\cdot)$ is uniformly continuous on H, and $E(u)$ tends to zero uniformly on $B(0, 2R)$ as $h \to 0$. In addition, DN and DE have analogous properties. Thus the first term in the above estimate is controlled. Moreover, N^h is uniformly continuous on

all of H, Φ^h converges uniformly to Φ on H, and so the second term in the above estimate tends to zero as h goes to zero. The bounds on $\|\eta\|$, $\|\eta_1\|$ are provided in Lemma 2.3. Hence, $\|\eta_1 - \eta\|$ may be made arbitrarily small uniformly in ρ by requiring h to be sufficiently small. We conclude, therefore that given and $\epsilon' > 0$ there is a $K(\epsilon') > 0$ such that $\|E_2\| \leq \epsilon'$ for all $h \leq K(\epsilon')$ and for all $p \in P_\Lambda H$ and $\rho \in P_\Lambda H : \|\rho\| = 1$.

For the last term, E_3, we use (2.7), (2.6), and Assumptions (G^h) to conclude that

$$\|E_3\| \leq \ell\|\rho - \rho^h\|$$
$$= \ell\|P_{\tilde{\Lambda}}\{[DG(\xi + \Phi(\xi)) - D\tilde{G}_h(\xi + \Phi(\xi))](I + D\Phi(\xi))\eta\}\|$$
$$\leq \sigma(\Lambda)K(R)\ell(1 + \ell)\phi^{-1}h.$$

Returning to (3.5), we conclude that given any $\epsilon' > 0$ there is a $h_0(\epsilon') > 0$ such that

$$\sup_{p \in P_\Lambda H} \|D\Phi(p) - D\Phi^h(p)\|_{op} \leq \epsilon'$$

for all $h \leq h_0$. ∎

4. CONSEQUENCES

While the persistence of the inertial manifold under C^1 perturbations is interesting in its own right, there are some consequences that are interesting to consider. The restriction of the map G to its inertial manifold gives the map for $p_n \in P_\Lambda H$

$$p_{n+1} = P_\Lambda G(p_n + \Phi(p_n)) \tag{4.1}$$

defined on the finite-dimensional space $P_\Lambda H$. This finite-dimensional system, called an inertial form, has the following properties:

• The inertial form has the same dynamics as the original map G.

• If the map G originates from the PDE as in (1.3), and if M is a bounded invariant set for the PDE, $P_\Lambda M$ is a bounded invariant set of the inertial form.

• The map $P_\Lambda G$ is at least C^1 on the space $P_\Lambda H$ which is finite dimensional.

Next we consider perturbations of the map G. Again the map G_h may be a fully discrete (space and time) approximation of (1.1). We will give an example in the next section. We need only assume Assumptions (G^h). That is, we assume that the map G_h is close to the map G in the C^1

norm, uniformly over some appropriately chosen ball containing the global attractor of the map G. The map $\tilde{G}_h$, (3.1) and restricted to its inertial manifold gives the system

$$p_{n+1} = P_\Lambda \tilde{G}_h(p_n + \Phi^h(p_n)). \tag{4.2}$$

This finite-dimensional system has the properties:

- Equation (4.2) has the same dynamics as the original map $\tilde{G}_h$ and agrees with the map G_h in the ball $B(0, R)$.

- The map $P_\Lambda \tilde{G}_h(p_n + \Phi^h(p_n))$ is at least C^1 on the space $P_\Lambda H$ which is finite dimensional.

- $P_\Lambda G(p + \Phi(p)) - P_\Lambda \tilde{G}_h(p + \Phi^h(p))$ may be made arbitrarily small in the C^1 norm on $B(0, R)$ by requiring h to be sufficiently small. See Theorem 3.2.

Since both (4.1) and (4.2) are finite-dimensional systems, one may apply results concerning the persistence of invariant sets under perturbation studied for finite-dimensional systems directly to the infinite-dimensional map G. Results that apply include those described in [3, 4, 12, 21, 26, 33, 34, 36]. Moreover, the bounded invariant sets of G reside in the ball $B(0, R)$. The above results indicate that the perturbed invariant sets converge to the true set for h sufficiently small. However, the map $\tilde{G}_h$ agrees with the map G_h inside $B(0, R)$. In particular,

Invariant sets that are known to persist under C^1 perturbation for the finite-dimensional case are captured by the map G_h for h sufficiently small.

5. AN EXAMPLE

We consider a dissipative PDE that may be expressed as an evolution equation in a Hilbert space H (as (1.1)) in the form

$$\frac{du}{dt} + Au + R(u) = 0, \tag{5.1}$$

$$u(0) = u_0.$$

We will assume that the linear operator A is an unbounded self-adjoint operator with compact inverse. We assume that the nonlinear term is C^1 and satisfies

$$|A^{\gamma-\beta}R(u)| \le M(\rho) \qquad \forall u \in D(A^\gamma) : |A^\gamma u| \le \rho.$$

$$|A^{\gamma-\beta}R'(u)\mu| \le M(\rho)|A^\gamma \mu| \qquad \forall u, \mu \in D(A^\gamma) : |A^\gamma u| \le \rho,$$

with $\gamma \geq 0$ and $\beta \in [0, 1)$ and where $M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a given monotonic increasing function. For convenience we set $\| \cdot \| := |A^\gamma \cdot |$. We suppose that the PDE is dissipative in the sense that there is an $R > 0$ such that for any solution of the PDE there is a $T^*(u_0) > 0$ such that

$$\|u(t)\| \leq R \qquad \forall t \geq T^*.$$

Thanks to the dissipative property of the PDE we may truncate the nonlinear term as we did for the map

$$F(u) := \theta_{2R}(\|u\|^2)R(u).$$

One may check that $F(u) = 0$ for $\|u\| \geq 4R$, $F(u) = R(u)$ for $\|u\| \leq \sqrt{8}R$, and

$$|A^{\gamma-\beta}F(u)| \leq K_1 \qquad \forall u \in \mathscr{D}(A^\gamma).$$

$$|A^{\gamma-\beta}F'(u)\mu| \leq K_2|A^\gamma\mu| \qquad \forall u, \mu \in \mathscr{D}(A^\gamma),$$

for some positive constants K_1, K_2 depending on R. The evolution equation

$$\frac{du}{dt} + Au + F(u) = 0 \tag{5.2}$$

has the same dynamics as the original PDE inside the absorbing set $B(0, R)$.

Remark. The reason for choosing the cut-off function θ_{2R} so that $F(u) = R(u)$ for $\|u\| \leq \sqrt{8}R$ is so that when we verify Assumptions (G^h) we need only check that the perturbation approximates the evolution equation (5.1) and not the prepared equation (5.2).

As in the Introduction we define

$$L(\tau) = e^{-A\tau}, \qquad N(\tau, u_0) = -\int_0^\tau e^{-(\tau-s)A}F(u(s))ds,$$

where $\tau > 0$ and $u(s)$ solves (5.2). Then

$$G(\tau, u_0) = L(\tau)u_0 + N(\tau, u_0) \tag{5.3}$$

and $G(\tau, u_0) = u(\tau)$.

We assume that the eigenvalues of A, $0 < \lambda_1 \leq \lambda_2 \ldots \leq \lambda_j \to \infty$, repeated with their multiplicities satisfy for any positive K_3, K_4

$$\lambda_m \geq K_3, \qquad \lambda_{m+1} - \lambda_m \geq K_4\lambda_{m+1}^\beta \tag{5.4}$$

for some $m \geq 0$. We set $P_\Lambda = P_m$ the projection onto the first m eigenfunctions of the operator A and $Q_m = I - P_m$. Because A is self adjoint, $\sigma(\Lambda) = 1$ in Assumptions (G). PDEs that are known to satisfy the

above assumptions include the Kuramoto–Sivashinsky equation, the complex Ginzburg–Landau equations, certain reaction-diffusion equations, as well as other PDEs. See the references mentioned in the introduction.

As shown in Jones and Stuart [24, Section 4], Assumptions (G) and Conditions (C2)–(C4) are satisfied for $\mu \in (e^{-1}, 1)$ and with

$$\tau = \frac{1}{\lambda_{m+1}}, \qquad b = e^{-\lambda_m \tau}, \qquad a = e^{-\lambda_{m+1}\tau}, \qquad c = e^{-\lambda_1 \tau},$$

$$B_2 = \tau^{1-\beta} K(K_1, K_2, \ell),$$

where $K(\cdot)$ is some positive constant depending on K_1, K_2, ℓ. The verification of (C1) follows exactly that of (C4). Indeed, we choose $K_4 = 4eK(K_1, K_2, \ell)(1 + \ell)$. (In the case we only require the inertial manifold to be Lipschitz, the multiplication of four would not be required.) With this choice for K_4 we multiply (5.4) by τ and use the inequality $e^x - 1 \geq x$ for positive x. We conclude that $e^{-\lambda_m \tau} - e^{-\lambda_{m+1}\tau} \geq 4\tau^{1-\beta}(1 + \ell)K(K_1, K_2, \ell)$, which is (C1).

We have

THEOREM 5.1. *Suppose that* (5.4) *is satisfied. Then there exists an m sufficiently large so that Assumptions* (G) *and Conditions* (C) *hold. For such m the map* $G(u) = Lu + N(u)$ *defined by* (5.3) *has an inertial manifold representable as the graph of a C^1 function* $\Phi : P_m H \to Q_m H$.

We turn to approximations of (5.1). We consider for simplicity a Galerkin approximation in the space variable and a semi-implicit time discretization. Finite element approximations are considered in Jones and Stuart [24], while finite difference approximations are studied in Jones [22] (see also Lord [29]). A Galerkin approximation as well as other ways to approximate the inertial manifold in the C^1 norm is studied in Jones and Titi [25]).

In order to simplify the error estimates we assume for convenience that the nonlinear term additionally satisfies

$$\sup_{\substack{\mu \in D(A^\gamma) \\ |A^\gamma \mu| = 1}} |A^{\gamma-\beta}(DF(u_1) - DF(u_2))\mu| \leq L_1 |A^\gamma(u_1 - u_2)| \qquad (5.5)$$

$$\sup_{\substack{\mu_1, \mu_2 \in D(A^\gamma) \\ |A^\gamma \mu_1| = |A^\gamma \mu_2| = 1}} |A^{\gamma-\beta}(D^2 F(u_1) - D^2 F(u_2))(\mu_1, \mu_2)|$$

$$\leq L_2 |A^\gamma(u_1 - u_2)|, \qquad (5.6)$$

for all $u_1, u_2 \in \mathscr{D}(A^\gamma)$. We emphasize that these assumptions are not necessary for our results to hold.

We approximate the solutions to (5.1) with

$$\frac{v_{n+1} - v_n}{\Delta t} + Av_{n+1} + P_N R(v_n) = 0, \tag{5.7}$$

where P_N is the projection onto the first N eigenfunctions of the operator A and $v_n \in P_N H$ for all $n \geq 0$. Theorem 5.1 requires that $\tau = 1/\lambda_{m+1}$ in the map G defined by (5.3). Therefore, we consider the map

$$G_{h,o}(v_0) = (I + \Delta t A)^{-1} v_0 - \Delta t \sum_{j=1}^{l} (I + \Delta t A)^{j-l-1} P_N R(v_{j-1}), \tag{5.8}$$

where $l\Delta t = T = \lambda_{m+1}^{-1}$. We define the perturbed map G_h by

$$G_h(u) = G_{h,o}(P_N u) \qquad \forall u \in H. \tag{5.9}$$

Left to verify are the two statements of Assumptions (G^h). Both are classical error estimates showing that solutions of the perturbed system G_h approximate the true solutions over finite-time intervals. In Jones and Stuart [24, Eq. (5.12)] such an error estimate is given, but for a discretization of the prepared equation (5.2). In the current setting we need only check the closeness of (5.9) to the solutions of (5.1) (see the remark above). However, rather than repeating the error analysis, we give the errors estimates given in Jones and Stuart [24]. The estimates are not as sharp as for direct approximation of (5.1). This is due to the fact that the only Lipschitz property the prepared nonlinear term satisfies is $|A^{\gamma-\beta}(F(u) - F(v))| \leq K|A^\gamma(u - v)|$ for $u, v \in D(A^\gamma)$.

We have from Jones and Stuart [24, Eq. (5.12)]

$$\|G(u) - G_h(u)\|$$
$$\leq K_\epsilon \lambda_{m+1} \left(\frac{\lambda_N}{\lambda_{m+1}}\right)^\epsilon \Delta t + K\left(\frac{\lambda_{m+1}}{\lambda_{N+1}}\right)^{1-\beta}, \qquad \forall u \in B(0, 2R), \tag{5.10}$$

where $K_\epsilon \to \infty$ as $\epsilon \to 0$. This estimate makes use of (5.5).

The second inequality in Assumptions (G^h) follows in the same way. Under our assumptions about the term $F(u)$ the Fréchet derivative of the solution to (5.2) with respect to the initial data exists and

$$\mu(t) = DS(t, u_0)\mu_0$$

satisfies

$$\frac{d\mu}{dt} + A\mu + DR(S(t)u_0)\mu = 0, \tag{5.11}$$

$$\mu(0) = \mu_0.$$

A similar situation holds for (5.8) and

$$DG_h(u_0)\mu_0 = (I + \Delta tA)^{-l}P_N\mu_0 - \sum_{j=1}^{l}(I + \Delta tA)^{j-l-1}P_N DR(P_N u_{j-1})\mu_j,$$

where the u_n are given by

$$u_n = (I + \Delta tA)^{-1}P_N u_0 - \Delta t\sum_{j=1}^{n}(I + \Delta tA)^{j-n-1}P_N R(P_N u_{j-1}).$$

Following the proof of (5.10) almost verbatim, we have that

$$\|DG(u) - DG_h(u)\|_{op}$$
$$\leq K_\epsilon\lambda_{m+1}\left(\frac{\lambda_N}{\lambda_{m+1}}\right)^\epsilon \Delta t + K\left(\frac{\lambda_{m+1}}{\lambda_{N+1}}\right)^{1-\beta}, \qquad \forall u \in B(0, 2R). \qquad (5.12)$$

This is where (5.6) is used. One may check that the map G_h is a diffeomorphism provided the CFL like condition $C_2\Delta t\lambda_N^\beta < 1$ is satisfied. We recall that C_2 depends on u as seen from the estimates on $R(u)$ below (5.1).

We therefore conclude with

THEOREM 5.2. *For m sufficiently large the map* G, (5.3), *has an inertial manifold representable as the graph of a C^1 function* Φ. *The finite-dimensional system*

$$p_{n+1} = P_m G(p_n + \Phi(p_n)) \qquad (5.13)$$

has the same dynamics as the map G; if M is a bounded invariant set of the PDE, $P_m M$ is a bounded invariant set of (5.13). For h (i.e., the right hand side of (5.10), (5.12)) sufficiently small, the map $\tilde{G}_h$ (defined in (3.1) from the map G_h) has an inertial manifold representable as the graph of a C^1 function Φ^h. *The finite-dimensional system*

$$p_{n+1} = P_m\tilde{G}_h(p_n + \Phi^h(p_n))$$

converges in the C^1 norm to (5.13) over the ball $B(0, R)$ as h tends to zero.

Those bounded invariant sets that are known to persist under C^1 perturbation reside in a neighborhood of the unperturbed set (see Pliss and Sell [33] for example). However, the map $p_{n+1} = P_m\tilde{G}_h(p_n + \Phi^h(p_n))$ agrees with the Galerkin spatial discretization and the semi-implicit Euler time discretization of (5.1) in the ball $B(0, R)$. Also the bounded invariants sets of $p_{n+1} = P_m G(p_n + \Phi(p_n))$ are the same as those of (5.1). We therefore conclude that the discretization (5.7) of (5.1) captures those bounded invariant sets that are known to persist under C^1 perturbation (in the context of finite-dimensional results) for h sufficiently small.

ACKNOWLEDGMENTS

REFERENCES

1. F. Alouges and A. Debussche, On the qualitative behavior of the orbits of a parabolic partial differential equations and its discretization in the neighborhood of a hyperbolic fixed point, *Numer. Funct. Anal. Optim.* **12** (1991), 253–269.

2. P. W. Bates, K. Lu, and C. Zeng, Existence and persistence of invariant manifolds for semiflows in Banach Space, preprint, 1997.

3. W.-J. Beyn, On invariant closed curves for one-step methods, *Numer. Math.* **51** (1987), 103–122.

4. W.-J. Beyn and J. Lorenz, Center manifolds of dynamical systems under discretization, *Numer. Funct. Anal. Optim.* **9** (1987), 381–414.

5. S.-N. Chow, K. Lu, and G. R. Sell, Smoothness of inertial manifolds, *J. Math. Anal. Appl.* **169** (1992), 283–321.

6. P. Constantin, C. Foias, B. Nicolaenko, and R. Temam, "Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations," Applied Mathematics Sciences, Vol. 70, Springer-Verlag, New York/Berlin, 1988.

7. P. Constantin, C. Foias, B. Nicolaenko, and R. Temam, Spectral barriers and inertial manifolds for dissipative partial differential equations, *J. Dynamics Differential Equations* **1** (1989), 45–73.

8. P. Constantin, C. Foias, and R. Temam, On the large time Galerkin approximation of the Navier–Stokes equations, *SIAM J. Numer. Anal.* **21** (1984), 615–634.

9. F. Demengel and J.-M. Ghidaglia, Some remarks on the smoothness of inertial manifolds, *Nonlinear Anal.* **16** (1991), 79–87.

10. F. Demengel and J. M. Ghidaglia, Time-discretization and inertial manifolds, *Math. Mod. Numer. Anal.* **23** (1989), 395–404.

11. E. Fabes, M. Luskin, and G. Sell, Construction of inertial manifolds by elliptic regularization, *J. Differential Equations* **89** (1991), 355–387.

12. N. Fenichel, Persistence and smoothness of invariant manifolds for flows, *Indiana Univ. Math. J.* **21** (1971), 193–226.

13. C. Foias, B. Nicolaenko, G. R. Sell, and R. Temam, Inertial manifolds for the Kuramoto–Sivashinsky equation and an estimate of their lowest dimensions, *J. Math. Pures Appl.* **67** (1988b), 197–226.

14. C. Foias, G. Sell, and R. Temam, Variétés inertielles des équations différentielles dissipatives, *C. R. Acad. Sci. Paris Sér. I* **301** (1985), 139–142.

15. C. Foias, G. Sell, and R. Temam, Inertial manifolds for nonlinear evolutionary equations, *J. Differential Equations* **73** (1988), 309–353.

16. C. Foias, G. Sell, and E. S. Titi, Exponential tracking and approximation of inertial manifolds for dissipative nonlinear equations, *J. Dynamics and Differential Equations* **1**, No. 2 (1989), 199–243.

17. C. Foias and E. S. Titi, Determining nodes, finite difference schemes and inertial manifolds, *Nonlinearity* **4** (1991), 135–153.

18. J. K. Hale, "Asymptotic Behavior of Dissipative Systems," Mathematical Surveys and Monographs, Vol. 25, Amer. Math. Soc., Providence, 1988.

19. D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Lecture Notes in Mathematics, Springer-Verlag, New York, 1981.

20. J. G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier–Stokes problems. Part II Stability of solutions and error estimates uniform in time, *SIAM J. Numer. Anal.* **21** (1986), 615–634.

21. M. W. Hirsh, C. C. Pugh, and M. Shub, "Invariant Manifolds," Lecture Notes in Mathematics, Vol. 583, Springer-Verlag, New York, 1977.

22. D. A. Jones, On the behavior of attractors under finite difference approximation, *Numer. Funct. Anal. Optim.* **16** (1995), 1155–1180.

23. D. A. Jones and S. Shkoller, Perisitence of invariant manifolds for nonlinear PDEs, *Stud. Appl. Math.*, in press.

24. D. A. Jones and A. M. Stuart, Attractive invariant manifolds under approximation: Inertial manifolds, *J. Differential Equations* **123** (1995), 588–637.

25. D. A. Jones and E. S. Titi, C^1 approximations of inertial manifolds for dissipative nonlinear equations, *J. Differential Equations* **126** (1996).

26. P. Kloeden and J. Lorenz, Stable attracting sets in dynamical systems and their one-step discretizations, *SIAM J. Numer. Anal.* **23** (1986), 986–995.

27. S. Larsson and J. M. Sanz-Serna, The behaviour of finite element solutions of semi-linear parabolic problems near stationary points, *SIAM J. Numer. Anal.* **31** (1994), 1000–1018.

28. S. Larsson and J. M. Sanz-Serna, A shadowing result with applications to finite element approximation of reaction-diffusion equations, preprint, 1996-05, *Dept. of Math.*, Chalmers Univ. of Tech., Göteborg, Sweden, 1996.

29. G. J. Lord, Attractors and inertial manifolds for finite difference approximations of the complex Ginzburg–Landau equation, *SIAM J. Numer. Anal.*, in press.

30. J. Mallet-Paret and G. Sell, Inertial manifolds for reaction diffusion equations in higher space dimensions, *J. Amer. Math. Soc.* **1** (1988), 805–866.

31. Y.-R. Ou and S. S. Sritharan, Analysis of regularized Navier–Stokes equations, II, *Quart. Appl. Math.* **49** (1991), 687–728.

32. A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York/Berlin, 1993.

33. V. A. Pliss and G. R. Sell, Perturbations of attractors of differential equations, *J. Differential Equations* **92** (1991), 100–124.

34. R. J. Sacker, A perturbation theorem for invariant manifolds and Hölder continuity, *J. Math. Mech.* **18** (1969), 705–762.

35. G. R. Sell and Y. You, Inertial manifolds: The non-self-adjoint case, *J. Differential Equations* **96** (1992), 203–255.

36. A. M. Stuart, Perturbation theory for infinite-dimensional dynamical systems, *in* "Theory and Numerics of Ordinary and Partial Differential Equations," (M. Ainsworth, J. Levesley, W. A. Light, and M. Marletta, Eds.), 1994.

37. R. Temam, "Infinite Dimensional Dynamical Systems in Mechanics and Physics," Springer-Verlag, New York, 1988.

38. E. S. Titi, On a criterion for locating stable stationary solutions to the Navier–Stokes equations, *Nonlinear Anal.* **11** (1987), 1085–1102.

39. E. S. Titi, Un critère pour l'approximation des solutions périodiques des équations de Navier–Stokes, *C.R. Acad. Sci. Paris Sér. I* **312**, No. 1 (1991), 41–43.