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Approximation of dissipative partial differential equations over long time intervals

Abstract In this article the numerical analysis of dissipative semilinear evolution equations with sectorial linear part is reviewed. In particular the approximation theory for such equations over long time intervals is discussed. Emphasis is placed on studying the effect of approximation on certain invariant objects which play an important role in understanding long time dynamics. Specifically the existence of absorbing sets, the upper and lower semicontinuity of global attractors and the existence and convergence of attractive invariant manifolds, such as the inertial manifold and unstable manifolds of equilibrium points, is studied.

1 Introduction

In this paper we consider initial value problems of the form

$$u_t = f(u), \quad u(0) = u_0. \tag{1.1}$$

In particular, our interest is in the approximation of the equation as $t \to \infty$. Recall that standard error estimates typically grow exponentially with the time interval under consideration and are hence of no direct value in this context. Our study will be focussed on partial differential equations but, to introduce the main ideas, we consider several illustrative examples in ordinary differential equations.

Examples

(i) *Dissipativity.* Consider the equation (1.1) with vector field $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(u) = u - u^3. \tag{1.2}$$

It is straightforward to show that

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 \leq 1 - |u|^2$$

and hence that

$$|u(t)|^2 \leq 1 + e^{-2t}[|u(0)|^2 - 1].$$

Thus there exists $T = T(|u(0)|, \epsilon)$ such that

$$|u(t)|^2 \leq 1 + \epsilon, \quad \forall t \geq T.$$

This shows that the solution satisfies an asymptotic bound which is independent of initial data. Many physical systems exhibit such a property and it is often a mathematical manifestation of some form of energy dissipation. In some cases it may be important to preserve the property under approximation.

The backward Euler method applied to (1.1), (1.2) yields the map

$$U^{n+1} = U^n + \Delta t[U^{n+1} - (U^{n+1})^3].$$

A little calculation shows that

$$|U^{n+1}|^2 \leq |U^n|^2 + 2\Delta t[1 - |U^{n+1}|^2]$$

and hence that

$$|U^{n+1}|^2 \leq (1 + 2\Delta t)^{-1}[|U^n|^2 + 2\Delta t].$$

Induction yields

$$|U^n|^2 \leq 1 + (1 + 2\Delta t)^{-n}[|U^0|^2 - 1].$$

Thus there exists $N = N(|U^0|, \epsilon)$ such that

$$|U^n|^2 \leq 1 + \epsilon, \quad \forall n \geq N.$$

This shows that the dissipativity of the equation is preserved under discretization for any $\Delta t > 0$.

In contrast, the explicit Euler scheme applied to (1.1),(1.2) is not dissipative: we obtain the map

$$U^{n+1} = U^n + \Delta t[U^n - (U^n)^3].$$

If we let

$$V = [1 + \frac{2}{\Delta t}]^{\frac{1}{2}}$$

and set $U^0 = V$, then the map admits the solution

$$U^n = (-1)^n V.$$

Since $|U^n| = |U^0|$, it is clear that the amplitude of the solution is not bounded independently of initial data for any n, however large; thus dissipativity does not occur for all fixed $\Delta t > 0$. Indeed, if $|U^0| > 1$, to obtain dissipativity for the explicit Euler scheme requires the restriction

$$\Delta t \leq \frac{2}{|U^0|^2 - 1}.$$

These two examples illustrate a general principle – only certain methods (such as backward Euler) will dissipate on the whole phase space for any $\Delta t > 0$. Most methods (such as forward Euler) will dissipate on any bounded set, but this requires a time-step restriction dependent upon the size of that set. In section 3 we shall discuss the dissipativity of numerical schemes. □

(ii) *The Global Attractor.* The notion of dissipativity observed in the previous example can be abstracted thus: a system is dissipative if there is a bounded set $\mathcal{B}$, independent of initial data, which all trajectories starting within any bounded set E enter and remain inside after a finite time $T = T(E, \mathcal{B})$. An analogous definition can be made for maps.

The set $\mathcal{B}$ is known as an *absorbing set.* For the equation (1.1), (1.2), and its backward Euler approximation, $\mathcal{B} = [-1-\epsilon, 1+\epsilon]$.

The global attractor $\mathcal{A}$ is found by mapping the set $\mathcal{B}$ forward under the equation (1.1) and seeing what remains as $t \to \infty$. A precise definition is given in section 4. For equation (1.1), (1.2) the global attractor is simply the interval $[-1, 1]$. This may be understood by noting that, since $f : \mathbb{R} \mapsto \mathbb{R}$, the flow is in gradient form as (1.2) may be written as

$$f(u) = -F'(u), \quad F(u) = \frac{1}{4}(1-u^2)^2.$$

Hence

$$\frac{d}{dt}\{F(u(t)\} = -u_t^2.$$

Thus $F(u)$ acts as a potential well for the equation (1.1), (1.2); since F has minima at $u = \pm 1$ and a maximum at $u = 0$ it follows that any bounded set in $\mathbb{R}$ is mapped into an ϵ neighbourhood of the interval $[-1, 1]$ in a finite time $T = T(\epsilon)$. Figure 1.1 shows the potential well F governing the flow of (1.1), (1.2).

In many situations it is of interest to understand the effect of approximation on the global attractor for (1.2). However, the global attractor may be very sensitive to perturbation and can undergo discontinuous shrinking under arbitrarily small perturbations. Consider (1.2) with vector field given by $f_\epsilon(u)$:

$$f_\epsilon(u) = \left\{ \begin{array}{cc} -(u+1)^3 + \epsilon, & u \leq -1 \\ \epsilon(u^3/2 - 3u/2), & -1 < u < 1 \\ -(u-1)^3 - \epsilon, & u \geq 1 \end{array} \right\} \tag{1.3}$$

This vector field is $C^1(\mathbb{R}, \mathbb{R})$ for every $\epsilon \geq 0$. Furthermore, using the gradient structure of the equation it is straightforward to show that the problem is dissipative with absorbing set $\mathcal{B} = [-1-\delta, 1+\delta]$ for any $\delta > 0$ and hence has an attractor $\mathcal{A}_\epsilon$. The gradient flow has potential $F(u)$ shown in Figure 1.2 for $\epsilon > 0$ and in Figure 1.3 for $\epsilon = 0$.

The important point to observe from these figures is that for every $\epsilon > 0$ the attractor

$$\mathcal{A}_\epsilon = \{0\}, \quad \epsilon > 0,$$

a single point whilst for $\epsilon = 0$

$$\mathcal{A}_0 = [-1, 1],$$

an entire interval. This shows that the attractor $\mathcal{A}_0$ is *upper-semicontinuous* with respect to $\epsilon > 0$ but it is not *lower-semicontinuous.* Although the perturbation induced by ϵ in this example is not directly analogous to a numerical approximation, it nonetheless indicates an important point – without strong assumptions it may be difficult to prove lower-semicontinuity of attractors with respect to perturbations of any kind, including those induced by numerical approximation. This will be clearly illustrated in section 4. Roughly speaking, the difficulty associated with the attractor $\mathcal{A}_0$ is its lack of hyperbolicity, or viewed another way the fact that it is not exponentially attracting. □

(iii) Attractive Invariant Manifolds. Since attractors may vary discontinuously under approximation, it is sometimes of interest to study objects which are more robust under

perturbation. An important example is an exponentially attractive invariant manifold. For the purposes of this article it is sufficient to think of an invariant manifold as a graph relating one subset (or projection) of the solution variables to another subset (or projection); in this context the invariance means that solutions starting on the graph remain on the graph for all time.

Consider the equations

$$\begin{aligned} p_t &= p - p^3, \quad p(0) = p_0, \\ q_t &= -q + 3p^2 - 2p^4, \quad q(0) = q_0. \end{aligned} \tag{1.4}$$

For (1.4), setting

$$w = q - p^2$$

yields

$$w_t = -w.$$

Thus the graph $w = 0$, that is $q = p^2$, is invariant for the equation and, furthermore, the set of points $q = p^2$ is exponentially attracting since $w(t) = \exp(-t)w(0)$. Thus $w = 0$ is an exponentially attracting invariant manifold.

In the context in which we are interested, attractive invariant manifolds are important since they are either contained within the attractor (*unstable manifolds*) or contain the attractor (*inertial manifolds*) - see section 5. In general, the exponential attraction of the manifolds in question ensures good stability properties under perturbation. □

In this paper we review the analysis of the effect of numerical approximation on dissipativity, attractors and certain attractive invariant manifolds. Most of the results appear elsewhere in the literature but some are presented in the context of partial differential equations for the first time. Furthermore, an overview of the subject is given which relates a variety of different topics concerned with numerical approximation over long-time intervals.

In section 2 we describe the mathematical setting for the partial differential equations studied and for their approximations. In section 3 we discuss discretization to preserve dissipativity. Section 4 contains a study of the upper and lower semicontinuity of the global attractor under numerical approximation. Section 5 is concerned with the upper and lower semicontinuity of attractive invariant manifolds; in particular unstable manifolds and inertial manifolds will be considered.

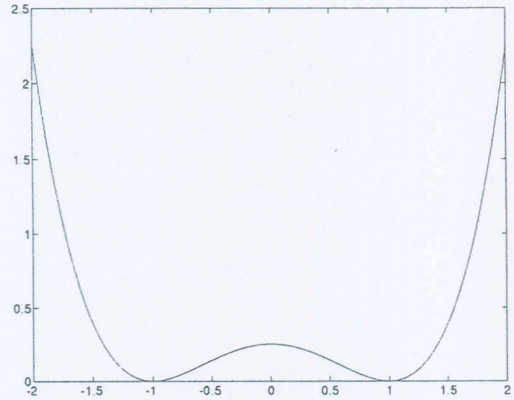

Figure 1.1: Potential Well for Equations (1.1),(1.2)

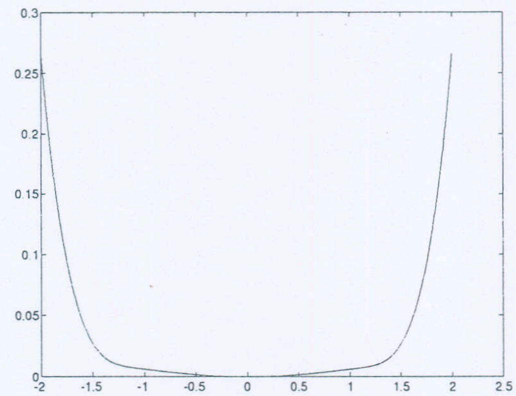

Figure 1.2: Potential Well for Equations (1.1), (1.3) with $\epsilon > 0$

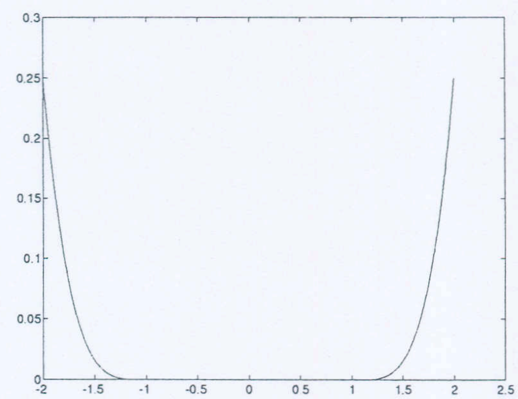

Figure 1.3: Potential Well for Equations (1.1), (1.3) with $\epsilon = 0$

FOR ALL FIGURES $F(u)$ IS ON THE VERTICAL AXIS AND u ON THE HORIZONTAL AXIS

2 Mathematical Setting

In the remainder of this paper we study the behaviour of the abstract evolution equation

$$\frac{du}{dt} + Au = F(u), \quad u(0) = u_0. \tag{2.1}$$

We consider (2.1) as an ordinary differential equation in a separable Hilbert space X with inner product $\langle \bullet, \bullet \rangle$ and norm $| \bullet |^2 = \langle \bullet, \bullet \rangle$. We assume that A is a densely defined sectorial operator with compact inverse, eigenvalues $\{\lambda_i\}$ and associated eigenfunctions $\{\varphi_i\}$. We also assume, without loss of generality, that A is positive definite and that the eigenvalues λ_i are ordered so that

$$0 < Re\{\lambda_1\} \leq Re\{\lambda_2\} \leq \ldots .$$

As in [39] we set $X^\alpha = D(A^\alpha)$ where $A^\alpha = (A^{-\alpha})^{-1}$ and for $\alpha > 0$

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt.$$

For $\alpha = 0$ we define $A^0 = I$. Then X^α is a Hilbert space with norm $|\bullet|_\alpha^2 = \langle A^{\alpha/2}\bullet, A^{\alpha/2}\bullet \rangle$. The operator A generates an analytic semigroup $L(t)$. (In section 5 we encounter the case where A is not positive definite in our study of unstable manifolds; in that case the spaces X^α and their associated norms are defined by considering the operator $\tilde{A} = A + \zeta I$ for $\zeta > 0$ chosen to make $\tilde{A}$ positive definite).

We assume that F satisfies conditions sufficient that (2.1) generates a semigroup $S(t)$ with the properties that

$$\left\{ \begin{array}{c} \exists S(t) : X^\gamma \mapsto X^\gamma, \quad \gamma \in (0,1), \text{ such that } u(t) = S(t)u_0, \\ \exists K = K(t,x,y) > 0 : |S(t)x - S(t)y|_\gamma \leq K|x-y|_\gamma, \\ \{S(t)u_0\}_{t \geq 0} \text{ is relatively compact in } X^\gamma. \end{array} \right\} \tag{2.2}$$

Thus (2.1) has solution $u(t) = S(t)u_0$ for every $u_0 \in X^\gamma$. Conditions on $F(\bullet)$ which yield (2.2) may be found, for example, in [23].

In the following we employ the notation

$$\mathcal{B}_\gamma(v,r) = \{u \in X^\gamma : |u - v|_\gamma < r\}.$$

Three examples, to which the theory considered in the remainder of the paper applies, are now described.

Examples

(i) *Reaction-Diffusion.*

$$\begin{aligned} u_t &= \Delta u + \lambda g(u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \\ g(u) &= \textstyle\sum_{j=1}^{2p-1} b_j u^j, \quad b_{2p-1} < 0. \end{aligned} \tag{2.3}$$

We take $X = L^2(\Omega)$ in this case. Existence of a Lipschitz continuous semigroup may be proved for $\gamma = \frac{1}{2}$ so that $S(t) : H_0^1(\Omega) \mapsto H_0^1(\Omega)$.

(ii) *Navier-Stokes equation in 2 Dimensions.*

$$
\begin{aligned}
u_t + u \cdot \nabla u = \tfrac{1}{R}\Delta u - \nabla p + h(x), \quad x \in \Omega,\\
\nabla \cdot u = 0, x \in \Omega, \\
u = 0, \quad x \in \partial\Omega.
\end{aligned}
\tag{2.4}
$$

Here X is the Hilbert space $\mathcal{H}$ comprising divergence free velocity fields contained in the space $L^2(\Omega)^2$ – see, for example, [44]. Existence of a Lipschitz continuous semigroup maybe proved for $\gamma = \frac{1}{2}$.

(iii) *Lorenz Equations.*

$$
\begin{aligned}
x_t &= -\sigma(x - y)\\
y_t &= rx - y - xz\\
z_t &= -bz + xy
\end{aligned}
\tag{2.5}
$$

Here $X = \mathbb{R}^3$ and, again, existence of a Lipschitz continuous semigroup may be proved for all $t \geq 0$. □

We shall consider the approximation of (2.1) in space by a finite difference, finite element or spectral method to yield the equation

$$
\frac{dU}{dt} + A^h U = F^h(U), \quad U(0) = U_0 \tag{2.6}
$$

for $U(t) \in \mathcal{V} \subset X$. Here $\mathcal{V}$ is a finite dimensional subspace of X.

For the temporal discretization we will consider the θ method in the form

$$
\frac{U^{n+1} - U^n}{\Delta t} + A^h U^{n+\theta} = F^h(U^{n+\theta}), \quad U(0) = U_0, \tag{2.7}
$$

where

$$
U^{n+\theta} := \theta U^{n+1} + (1-\theta)U^n, \quad \theta \in [0,1].
$$

We assume that A^h defines a sectorial operator on $\mathcal{V}$ and note that we may then define $X^{h,\alpha}$ to be the Hilbert space comprised of elements of $\mathcal{V}$ with norm $| \bullet |^2 = \langle A^{h,\alpha/2}\bullet, A^{h,\alpha/2}\bullet\rangle$. We employ the following notation:

$$
\mathcal{B}_{h,\alpha}(v,r) = \{u \in X^{h,\alpha} : |u - v|_{h,\alpha} < r\}.
$$

We also assume that (2.6) and (2.7) generate discrete Lipschitz continuous semigroups $S^h(t) : \mathcal{V} \mapsto \mathcal{V}$ and $S_n^{h,\Delta t} : \mathcal{V} \mapsto \mathcal{V}$ respectively.

3 Dissipativity

We start by making a precise definition of dissipativity, motivated by the Example (i) in section 1.

Definition 3.1 *Equation* (2.1) *(resp.* (2.6), (2.7)*) is said to be* dissipative *in* $X^\beta, \beta \in [0, \infty)$, *(resp.* $X^{h,\beta}$*) if there exists* $\rho > 0$ *such that for any* $r > 0$ *there is a* $T = T(\rho, r)$ *(resp.* $N = N(\rho, r)$*) such that*

$$S(t)\mathcal{B}_\gamma(0, r) \subseteq \mathcal{B}_\beta(0, \rho) \ \ \forall t > T$$

(resp.

$$S^h(t)\mathcal{B}_{h,\gamma}(0, r) \subseteq \mathcal{B}_{h,\beta}(0, \rho) \ \ \forall t > T$$

$$S_n^{h,\Delta t}\mathcal{B}_{h,\gamma}(0, r) \subseteq \mathcal{B}_{h,\beta}(0, \rho) \ \ \forall n > N.)$$

Remarks
(i) The notion of dissipativity for ordinary differential equation is discussed, for example, in [40] and [46]. Generalizations to partial differential equations may be found in [20] and [44].
(ii) For the finite dimensional approximations (2.6) and (2.7) it is clear that, for any fixed $h > 0$, all the spaces $X^{h,\alpha}$ are equivalent for all $\alpha > 0$ so that dissipativity in one space implies dissipativity in them all; however, in most cases, interest is in deriving discrete absorbing sets that have radii ρ bounded independently of $h \to 0$, for any given $\alpha > 0$.
(iii) In many applications of the partial differential equation the initial data is taken in $\mathcal{B}_0(0, r)$; however, the smoothing properties induced by e^{-At} often means that it is equivalent to take data in $\mathcal{B}_\gamma(0, r)$ for some $\gamma \in (0, 1)$ – see [23], [44].
(iv) Frequently dissipativity is proved initially in X. Use of the uniform Gronwall lemma [44] or, for certain gradient systems, a Lyapunov function [20] enables this to be extended to dissipativity in X^β for some $\beta > 0$. □

Throughout this section we assume that A is self-adjoint. We consider the case where $\gamma = \frac{1}{2}$ and

$$f(u) := F(u) - Au \tag{3.1}$$

satisfies the structural assumption

$$\exists a, b > 0 : \langle f(u), u \rangle \leq a - b|u|^2, \quad \forall u \in X^{\frac{1}{2}} \subset X. \tag{3.2}$$

Note that, since $X^{\frac{1}{2}} = D(A^{\frac{1}{2}})$, it follows that $\langle Au, u \rangle = |A^{\frac{1}{2}}u|^2$ is well-defined for $u \in X^{\frac{1}{2}}$ in the weak sense. There are many examples of equations satisfying (3.2) described in, for example, [44]. In particular the three examples described in section 2 all satisfy (3.2). Under (3.1), (3.2) it follows from (2.1) that

$$\frac{1}{2}\frac{d}{dt}|u|^2 \leq a - b|u|^2$$

and application of the Gronwall lemma shows that (2.1) is dissipative in X with

$$\rho = [(a + \epsilon)/b]^{\frac{1}{2}} \tag{3.3}$$

for any $\epsilon > 0$.

Under approximation in space, equation (2.1) yields (2.6), where now we define

$$f^h(u) := F^h(u) - A^h u. \tag{3.4}$$

In some applications it is of interest to perform the spatial approximation in such a way that $f^h(\bullet)$ satisfies a structural assumption analogous to (3.2), namely that

$$\exists a, b > 0 : \langle f^h(u), u\rangle \leq a - b|u|^2, \quad \forall u \in \mathcal{V} \subset X. \tag{3.5}$$

Note that, without loss of generality, we have assumed that the same constants a and b appear in (3.2) as in (3.5). The question of spatial discretization to retain the dissipativity of the underlying problem has been addressed by several authors. The earliest work in this direction appears to be [42], [31], [16], [15]. In [42] the dissipative properties of a Legendre-Galerkin approximation to a reaction-diffusion equation (2.3) were studied in both $L^2(\Omega)$ and $H_0^1(\Omega)$; however the restrictions on the discretization parameters required in this analysis are *initial data dependent* and it is natural to seek schemes for which this is not required. In [15] a finite difference scheme is constructed for the Kuramoto-Sivishinsky equation which preserves a condition closely related to (3.2) *without* initial data dependent restrictions on the discretization parameters; it is interesting to note that the work of [15] employs the same energy conserving approximation to uu_x analysed by Fornberg in [19]. In addition to proving dissipativity of the scheme, the paper [15] also studies conditions under which numerical approximations will blow up if energy-conserving discretizations of uu_x are not used; related issues for the viscous Burger's equations have been studied in [11]. The dissipative properties of certain nonlinear Galerkin methods are studied in [31] and [16]. In [13] analogous properties to those derived in [15] are proved for finite difference and finite element methods applied to a reaction-diffusion equation and in [37] for the Ginzburg-Landau equation. The dissipativity of certain finite element methods for the Navier-Stokes equation is studied in [2] and spectral methods for a nonlinear convection-diffusion equation in [26].

Henceforth we assume that spatial discretization has been performed in such a way that (3.5) holds. It is then natural to study the effect of time-discretization on dissipativity. The following result is of interest in this context. We employ the notation ρ given by (3.3) and define

$$R = \sup_{|v| \leq \rho} |f^h(v)|.$$

Recall that the true absorbing set has radius ρ. Note that R may be unbounded as $h \to 0$ since $f^h(\bullet) = F^h(\bullet) - A^h\bullet$ and A^h approximates an operator A with domain $D(A) \subset X$.

Theorem 3.1 Dissipativity *Consider application of* (2.7) *to* (2.6) *under the structural assumption* (3.4), (3.5). *Then* (2.7) *is dissipative in* X *for every* $\theta \in [\frac{1}{2}, 1]$; *furthermore, the absorbing set* $\mathcal{B}_0(0, \sigma)$ *has radius* σ *given by*

$$\sigma = \left\{ \begin{array}{c} \rho + \frac{\Delta t}{2} R, \quad \theta = \frac{1}{2}, \\ \rho + \Delta t(1-\theta)R, \quad \theta \in (\frac{1}{2}, 1), \quad \Delta t \in (0, \frac{2\rho}{(2\theta-1)R}] \\ \frac{1}{2\theta - 1}\rho, \quad \theta \in (\frac{1}{2}, 1), \quad \Delta t \in (\frac{2\rho}{(2\theta-1)R}, \infty) \\ \rho, \quad \theta = 1. \end{array} \right\} \tag{3.6}$$

Proof. We have

$$\frac{U^{n+1}-U^n}{\Delta t}=f(U^{n+\theta}), \tag{3.7}$$

where f satisfies (3.5). Since

$$U^{n+\theta}=\frac{U^{n+1}+U^n}{2}+(\theta-\frac{1}{2})(U^{n+1}-U^n) \tag{3.8}$$

the inner product with $U^{n+\theta}$ yields, for $\theta\in[\frac{1}{2},1]$,

$$\frac{|U^{n+1}|^2-|U^n|^2}{2\Delta t}\le a-b|U^{n+\theta}|^2. \tag{3.9}$$

Let

$$\sigma_1=\rho+\Delta t(1-\theta)R,\quad \sigma_2=\frac{\rho}{(2\theta-1)}. \tag{3.10}$$

We show that $\mathcal{B}_0(0,\sigma_1)$ is an absorbing set for all $\theta\in[\frac{1}{2},1]$ and that $\mathcal{B}_0(0,\sigma_2)$ is absorbing for $\theta\in(\frac{1}{2},1]$. The result then follows - for $\theta\in(\frac{1}{2},1)$ the minimum of σ_1,σ_2 has been chosen for each $\Delta t\in(0,\infty)$.

First consider $\mathcal{B}_0(0,\sigma_1)$. Let $\epsilon>0$. It is clear that either

$$a-b|U^{n+\theta}|^2\le-\epsilon, \tag{3.11}$$

or

$$a-b|U^{n+\theta}|^2\ge-\epsilon. \tag{3.12}$$

If (3.12) holds then we have

$$|U^{n+\theta}|\le\rho$$

and, since $U^{n+\theta}=U^{n+1}+(\theta-1)(U^{n+1}-U^n)$, we have $|U^{n+1}|\le\sigma_1$. Now, if (3.11) holds and if $|U^n|\le\sigma_1$ then, by (3.9), we have $|U^{n+1}|\le|U^n|\le\sigma_1$. These two observations show that $\mathcal{B}_0(0,\sigma_1)$ is positively invariant since, under either (3.11) or (3.12) we have $|U^n|\le\sigma_1\Rightarrow|U^{n+1}|\le\sigma_1$. Furthermore, it follows from (3.9), (3.11) that iterates starting in any bounded set containing $\mathcal{B}_0(0,\sigma_1)$ enter $\mathcal{B}_0(0,\sigma_1)$ in a finite number of steps. Thus we have exhibited an absorbing set $\mathcal{B}_0(0,\sigma_1)$.

Now consider $\mathcal{B}_0(0,\sigma_2)$. Since $U^{n+\theta}=\theta U^{n+1}+(1-\theta)U^n$, equation (3.9) yields

$$\begin{aligned}\frac{|U^{n+1}|^2-|U^n|^2}{2\Delta t}&\le a-b[\theta^2|U^{n+1}|^2+2\theta(1-\theta)\langle U^n,U^{n+1}\rangle+(1-\theta)^2|U^n|^2]\\ &\le a-b[\theta^2|U^{n+1}|^2-\theta(1-\theta)|U^{n+1}|^2-\theta(1-\theta)|U^n|^2+(1-\theta)^2|U^n|^2]\\ &\le a-b[\theta(2\theta-1)|U^{n+1}|^2+(1-\theta)(1-2\theta)|U^n|^2].\end{aligned}$$

Hence

$$[1+2\Delta tb\theta(2\theta-1)]|U^{n+1}|^2\le[1-2\Delta tb(1-\theta)(1-2\theta)]|U^n|^2+2a\Delta t. \tag{3.13}$$

Algebra shows that, for $\theta\in(\frac{1}{2},1]$,

$$\frac{|1-2\Delta tb(1-\theta)(1-2\theta)|}{1+2\Delta tb\theta(2\theta-1)|}<1,\quad\forall\Delta t>0.$$

Thus applying the Gronwall lemma to (3.13) yields

$$\limsup_{n\to\infty} |U^n|^2 \le \frac{a}{b(2\theta - 1)^2}$$

and it follows that $\mathcal{B}_0(0, \sigma_2)$ is absorbing. This completes the proof. $\square$.

Remarks

(i) Note that, for $\theta = 1$ the absorbing set has radius identical to that arising in the continuous case. For $\theta = \frac{1}{2}$ the dissipaitivty is rather weak since, because of the dependence of R on h, the absorbing set depends on $\Delta t/h^p$ for some $p > 0$. Thus a form of Courant restriction is required to make the absorbing set mesh independent. For example, if we consider the reaction-diffusion equation (2.3) and a standard finite element approximation based on piecewise linear triangulation of Ω, then $p = 2$.

(ii) In [27] the two-step backward differentiation formula is analysed for problems satisfying (3.2) and shown to be dissipative. For gradient systems it is possible to find second order time-accurate schemes which preserve dissipativity, without requiring a Courant restriction, by exploiting a discrete version of a Lyapunov function; see [13], where a modification of the Crank-Nicolson method constructed in [12] and the two-step backward differentiation formula are analysed in this context.

(iii) Theorem 3.1 is a synthesis of the ideas contained in [13], [28] and [2]. In [13] the dissipativity of the backward Euler scheme is studied for reaction-diffusion equations; in that paper a discrete version of the uniform Gronwall lemma is proved and employed to establish dissipativity in $H_0^1(\Omega)$ as well as $L^2(\Omega)$. In [28] Runge-Kutta methods are studied for ordinary differential equations (1.1) satisfying (3.2); it is proved that irreducible algebraically stable methods preserve dissipativity, making a connection with the exsiting theory of contractive methods derived in [6]. In [2] it was observed that, in the context of the Navier-Stokes equations (2.4), a linearization of the θ–method for $\theta \in (\frac{1}{2}, 1]$ has an absorbing set independent of $\Delta t > 0$. This interesting fact has been generalized slightly here and incorporated into our proof. A similar analysis for convection-diffusion equations may be found in [26].

(iv) The linearisation of the θ–method considered in [2] is of some practical interest since it allows the direct implementation of a dissipative numerical method involving only the inversion of *linear systems* at each step. This approach can be used in cases where $F(\bullet)$ may be represented as the sum of a bilinear form and a forcing function so that

$$F(v) = B(v, v) + g$$

and the numerical approximation $F^h(\bullet)$ may be represented as the sum of a bilinear form and a forcing function

$$F^h(v) = B^h(v, v) + g.$$

It is then possible to consider the time-stepping scheme

$$\frac{U^{n+1} - U^n}{\Delta t} + A^h U^{n+\theta} = B^h(U^n, U^{n+\theta}) + g, \quad U(0) = U_0.$$

If B satisfies the property

$$\langle v, B(w, v)\rangle = 0 \ \forall v, w \in X^{\frac{1}{2}}$$

(as in the Navier-Stokes equations, for example) and a spatially discrete analogue of this condition also holds then it is possible to generalise the analysis of Theorem 3.1 to cope with this case – see [2] for details. □

This concludes our analysis of dissipativity. We remark that the dissipativity of linear multistep methods is still an open question, as is the matter of the effect of the nonlinear solver on implicit methods such as (2.7). We also note that we have asked here for dissipativity of numerical schemes on the *whole phase space* $\mathcal{V}$ for fixed $h, \Delta t$; this is an extremely strong condition. In many cases we might expect to obtain dissipativity on any compact set $E \subset \mathcal{V}$ for $\Delta t, h$ sufficiently small in terms of the size of E. Such results have been proved for Runge-Kutta methods applied to the ordinary differential equation (1.1) under (1.3) in [29], [30], for nonlinear Galerkin methods in [10] and [41] and for spectral approximation of reaction diffusion convection equations in [26]; however there are still many open questions in this area for the approximation of partial differential equations.

4 The Global Attractor

In this section we define the global attractor for (2.1), (2.6) and (2.7) and then study the relationship between the true and approximate attractors. We employ the notation

$$\text{dist}_\beta(u, A) = \inf_{v\in A} |u - v|_\beta$$

$$\text{dist}_\beta(B, A) = \sup_{u\in B} \text{dist}_\beta(u, A)$$

$$\mathcal{N}_\beta(A, \epsilon) = \{u \in X^\beta : \text{dist}_\beta(u, A) < \epsilon\}$$

$$\mathcal{N}_{h,\beta}(A, \epsilon) = \{u \in X^{h,\beta} : \text{dist}_{h,\beta}(u, A) < \epsilon\}.$$

Notice that $\text{dist}_\beta(B, A) = 0$ is equivalent to the statement $\bar{B} \subseteq \bar{A}$ so that "dist" defines a semidistance – the asymmetric Hausdorff semidistance. We will also find the concept of an *invariant set* useful. A set B is said to be invariant under $S(t)$ if $S(t)B \equiv B$ for all $t \geq 0$.

Definition 4.1 *A set $\mathcal{A} \in X^\gamma$ (resp. $X^{h,\gamma}$) is said to* attract *a set $U \in X^\gamma$ (resp. $X^{h,\gamma}$) under a semigroup $S(t)$ (resp. $S^h(t), S_n^{h,\Delta t}$) if, for any $\epsilon > 0$ $\exists T = T(\epsilon)$ (resp. $N = N(\epsilon)$) such that*

$$S(t)U \in \mathcal{N}_\gamma(\mathcal{A}, \epsilon) \;\; \forall t \geq T,$$

(resp.

$$S^h(t)U \in \mathcal{N}_{h,\gamma}(\mathcal{A}, \epsilon) \;\; \forall t \geq T,$$

$$S_n^{h,\Delta t}U \in \mathcal{N}_{h,\gamma}(\mathcal{A}, \epsilon) \;\; \forall n \geq N.)$$

An attractor *is a compact invariant set which attracts an open neighbourhood of itself. An attractor $\mathcal{A}$ is said to be a* global attractor *if it attracts every bounded set $U \in X^\gamma$ (resp. $U \in X^{h,\gamma}$.)*

We now show how the existence of an absorbing set, together with some compactness, gives the existence of a global attractor [44]. To formalise this idea we require the following definition:

Definition 4.2 *The $\omega-$limit set of a bounded set $U \in X^\gamma$ (resp. $X^{h,\gamma}$) for $S(t)$ (resp. $S^h(t), S_n^{h,\Delta t}$) is defined by*

$$\omega(U) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)U},$$

(resp.

$$\omega(U) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S^h(t)U},$$

$$\omega(U) := \bigcap_{m\geq 0} \overline{\bigcup_{n\geq m} S_n^{h,\Delta t}U}.)$$

The following theorem is proved in [44]:

Theorem 4.1 *If $S(t)$ (resp. $S^h(t), S_n^{h,\Delta t}$) has an absorbing set $\mathcal{B}_\gamma(0,\rho)$ (resp. $\mathcal{B}_{h,\gamma}(0,\rho)$) and if $\exists t_c > 0$ (resp. $n_c > 0$) such that $\{S(t)E\}_{t\geq t_c}$ (resp. $\{S^h(t)E\}_{t\geq t_c}, \{S_n^{h,\Delta t}E\}_{n\geq n_c}$) is relatively compact for any bounded set $E \in X^\gamma$ (resp. $X^{h,\gamma}$), then $\omega(\mathcal{B}_\gamma(0,\rho))$ (resp. $\omega(\mathcal{B}_{h,\gamma}(0,\rho))$ is a global attractor for $S(t)$ (resp. $S^h(t), S_n^{h,\Delta t}$).*

Remarks

(i) In [44] Theorem 4.1 is used to construct attractors for $S(t) : X \mapsto X$ by finding absorbing sets in X and $X^\beta, \beta > 0$. In [20] the global attractor for gradient systems is constructed directly by using a Lyapunov function and the existence of absorbing sets is a consequence of the existence of an attractor or, alternatively, can be deduced from the existence of a Lyapunov function.

(ii) In finite dimensions relative compactness is automatic and the existence of an absorbing set in X implies the existence of an attractor. This has been used in a number of cases to construct global attractors for numerical schemes – see [42], [15], [13], [37] and [28]; in addition, the papers [42], [13] and [37] also prove $h-$ independent estimates on the size of the attractor in $X^{h,\beta}$ for some $\beta > 0$. Furthermore, discrete Lyapunov functions can also be used to construct a global attractor for the numerical approximation of gradient systems; see [13].

(iii) In many cases it is not possible to show that an approximate scheme has a *global* attractor as the example of the forward Euler method applied to (1.1), (1.2) shows; however, the work of [22] shows that *local* attractors can be constructed for the approximation under reasonable hypotheses. □

For the remainder of this section we assume that it has been established that (2.1), (2.6), (2.7) have global attractors $\mathcal{A}, \mathcal{A}^h, \mathcal{A}^{h,\Delta t}$ respectively and that, furthermore, these sets may be constructed as $\omega-$limit sets of absorbing sets:

$$\begin{gathered}\exists \beta \in [\gamma, 1), \ \textit{absorbing sets} \ \mathcal{B}_\beta(0,\rho), \ \mathcal{B}_{h,\beta}(0,\rho) \ \textit{for} \ S(t), S^h(t), S_n^{h,\Delta t} : \\ \mathcal{A} = \omega(\mathcal{B}_\beta(0,\rho)), \ \mathcal{A}^h = \omega(\mathcal{B}_{h,\beta}(0,\sigma)), \ \mathcal{A}^{h,\Delta t} = \omega(\mathcal{B}_{h,\beta}(0,\sigma)).\end{gathered} \tag{4.1}$$

We assume also that $\mathcal{B}_{h,\beta}(0,\sigma)$ is bounded in X^γ so that,

$$\exists r = r(\sigma) > 0 : U \in \mathcal{B}_{h,\beta}(0,\sigma) \Rightarrow U \in \mathcal{B}_\gamma(0,r). \tag{4.2}$$

In the light of our preceding remarks we know that these assumptions may be quite strong as only local attractors may exist for the approximate semigroup; however, since our aim is to convey the essential ideas, we proceed in this framework. These ideas can be modified to allow for the case where $\mathcal{A}, \mathcal{A}^h$ or $\mathcal{A}^{h,\Delta t}$ are only local attractors as in [22].

Our aim is to study the relationship between $\mathcal{A}$ and $\mathcal{A}^h$ or $\mathcal{A}$ and $\mathcal{A}^{h,\Delta t}$. ¿From the example in section 1 we expect that it may be possible to show that $\text{dist}_\gamma(\mathcal{A}^h, \mathcal{A})$ is small (upper-semicontinuity) but that showing that $\text{dist}_\gamma(\mathcal{A}, \mathcal{A}^h)$ is small (lower-semicontinuity) will not, in general, be possible. The following theorem concerns upper-semicontinuity.

Theorem 4.2 Upper-Semicontinuity of Attractors *Assume that:*

(I) equations (2.1), (2.6) *have global attractors $\mathcal{A}$ and $\mathcal{A}^h$ given by* (4.1) *and that* (4.2) *holds;*

(II) for any $T, \delta, \sigma > 0$ $\exists H = H(T,\delta,\sigma) > 0$ such that

$$|S(t)U - S^h(t)U|_\gamma \leq \delta, \quad T \leq t \leq 2T$$

for all $U \in \mathcal{B}_{h,\beta}(0,\sigma)$ and $h \in (0,H)$.

Then $\text{dist}_\gamma(\mathcal{A}^h, \mathcal{A}) \to 0$ as $h \to 0$.

Proof. Recall that, since $\mathcal{A}^h = \omega(\mathcal{B}_{h,\beta}(0,\sigma))$, it is sufficient to show that for any $\epsilon > 0$ $\exists T > 0$ and $H > 0$ such that, if $h \in (0,H)$, then $S^h(t)\mathcal{B}_{h,\beta}(0,\sigma) \subset \mathcal{N}_\gamma(\mathcal{A},\epsilon)$ $\forall t \geq T$.

Now, by (4.2) we have that $\mathcal{B}_{h,\beta}(0,\sigma) \subseteq \mathcal{B}_\gamma(0,r)$. Thus, for any $\epsilon > 0$ $\exists T > 0$:

$$S(t)\mathcal{B}_{h,\beta}(0,\sigma) \subset \mathcal{N}_\gamma(\mathcal{A},\epsilon/2) \ \forall t \geq T, \tag{4.3}$$

since $\mathcal{A}$ is the global attractor for $S(t)$ in X^γ. Furthermore, by assumption II of the theorem, for any $\epsilon > 0$ $\exists H > 0$:

$$\text{dist}_\gamma(S^h(t)\mathcal{B}_{h,\beta}(0,\sigma), S(t)\mathcal{B}_{h,\beta}(0,\sigma)) \leq \epsilon/2, \quad T \leq t \leq 2T, \tag{4.4}$$

for $h \in (0,H)$. Equations (4.3) and (4.4) show that, provided $h \in (0,H)$,

$$S^h(t)\mathcal{B}_{h,\beta}(0,\sigma) \subset \mathcal{N}_\gamma(\mathcal{A},\epsilon), \quad T \leq t \leq 2T.$$

For the purposes of induction assume that, provided $h \in (0,H)$

$$S^h(t)\mathcal{B}_{h,\beta}(0,\sigma) \subset \mathcal{N}_\gamma(\mathcal{A},\epsilon), \quad kT \leq t \leq (k+1)T \tag{4.5}$$

and note that we have proved this for $k = 1$. Since $S^h(kT)\mathcal{B}_{h,\beta}(0,\sigma) \subseteq \mathcal{B}_{h,\beta}(0,\sigma)$ (by definition of an absorbing set) we may repeat the arguments above (with the role of $t = 0$ taken by $t = kT$) to deduce that

$$\text{dist}_\gamma(S^h(t)\mathcal{B}_{h,\beta}(0,\sigma), S(t)\mathcal{B}_{h,\beta}(0,\sigma)) \leq \epsilon/2, \quad (k+1)T \leq t \leq (k+2)T$$

for $h \in (0, H)$. Hence by (4.3) we deduce that, provided that $h \in (0, H)$ we have

$$S^h(t)\mathcal{B}_{h,\beta}(0,\sigma) \subset \mathcal{N}_\gamma(\mathcal{A},\epsilon), \quad (k+1)T \leq t \leq (k+2)T. \tag{4.6}$$

Since (4.5) implies (4.6), induction on k shows that, for any $\epsilon > 0$ $\exists T > 0, H > 0$ such that

$$S^h(t)\mathcal{B}_{h,\beta}(0,\sigma) \subset \mathcal{N}_\gamma(\mathcal{A},\epsilon), \quad t \geq T$$

provided that $h \in (0, H)$. Since $\epsilon > 0$ is arbitrary, this yields the convergence result. □

Remarks

(i) Theorem 4.2 was first proved in [22] in a more general form where the existence of an approximate global attractor was not assumed but, rather, an approximate local atttractor constructed during the course of the proof.

(ii) Related results may be found in [33] where the weaker concept of uniformly asymptotically stable sets was studied for ordinary differential equations and in [34] where the method was generalized to partial differential equations. The paper [25] relates the approach of [33] to that of [22]. The book [44] also contains a result similar to Theorem 4.2.

(iii) In [42] the upper semicontinuity of approximate attractors was considered for (2.3) under Legendre-Galerkin approximation; in [35] such an analysis is considered for the finite element method applied to (2.3). The upper semicontinuity of attractors for the ordinary differential equations (1.1), (1.3) under approximation by Runge-Kutta methods is studied in [28]. For q-step linear multistep methods the issues are somewhat more complicated since the natural phase space for the temporal approximation is $\{X^\gamma\}^q$. This issue is considered in [24] where the existence and upper semicontinuity of attractors for strictly $A(\alpha)$-stable multistep methods is proved.
(iv) Note that, depending upon the value of γ, the error estimate in assumption (II) of Theorem 4.2 may be required for non-smooth initial data. The issue of non-smooth error estimates, and the relationship to attractor convergence, is considered in [35]. For finite difference methods the derivation of such non-smooth error estimates is considered for the Navier-Stokes equations in [45]. In [37] the use of non-smooth error estimates for finite difference approximations of the complex Ginzburg-Landau equation is avoided by deriving discrete regularity results for the approximate schemes which essentially enable β to be chosen arbitrarily large.
(v) Note that a result similar to Theorem 4.2 can be proved which incorporates the effect of time-discretization; see [22]. □

We now proceed to discuss the lower-semicontinuity of attractors under numerical approximation. As the example in section 1 shows this will not, in general, be possible unless strong hyperbolicity conditions are imposed upon the attractor. In order to make these conditions precise we need some further definitions. In the following we use dF to denote the Fréchet derivative of F.

Definition 4.3 *An* equilibrium point $\bar{u} \in X^\gamma$ *of* (2.1) *(resp.* $\bar{U} \in X^{h,\gamma}$ *of* (2.6), (2.7)*) satisfies*

$$A\bar{u} = F(\bar{u}),$$

(resp.

$$A^h\bar{U} = F^h(\bar{U}).)$$

The unstable manifold *of $\bar{u}$ (resp. $\bar{U}$) denoted by $W^u(\bar{u})$ (resp. $W^u_h(\bar{U}), W^u_{h,\Delta t}(\bar{U})$) is the set*

$$\{u_0 \in X^\gamma : (2.1) \text{ has a solution } \forall t \leq 0 \;\&\; u(t) \to \bar{u} \text{ as } t \to -\infty\},$$

(resp.

$$\{U_0 \in X^{h,\gamma} : (2.6) \text{ has a solution } \forall t \leq 0 \;\&\; U(t) \to \bar{U} \text{ as } t \to -\infty\},$$

$$\{U_0 \in X^{h,\gamma} : (2.7) \text{ has a solution } \forall n \leq 0 \;\&\; U^n \to \bar{U} \text{ as } n \to -\infty\}.)$$

The local unstable manifold *of $\bar{u}$ (resp. $\bar{U}$) denoted by $W^u(\bar{u};\epsilon)$ (resp. $W^u_h(\bar{U};\epsilon)$, $W^u_{h,\Delta t}(\bar{U};\epsilon)$) is the set*

$$\{u_0 \in W^u(\bar{u}) : u(t) \in \mathcal{B}_\gamma(\bar{u},\epsilon) \;\; \forall t \leq 0\},$$

(resp.

$$\{U_0 \in W^u_h(\bar{U}) : U(t) \in \mathcal{B}_{h,\gamma}(\bar{U},\epsilon) \;\; \forall t \leq 0\},$$

$$\{U_0 \in W^u_{h,\Delta t}(\bar{U}) : U^n \in \mathcal{B}_{h,\gamma}(\bar{U},\epsilon) \;\; \forall n \leq 0\}.)$$

An equilibrium point $\bar{u}$ of (2.1) *(resp. $\bar{U}$ of* (2.6), (2.7)*) is said to be hyperbolic if the spectrum of the linear operator $A - dF(\bar{u}) : D(A) \mapsto X$ (resp. $A^h - dF^h(\bar{U}) : \mathcal{V} \mapsto \mathcal{V}$) contains no points on the imaginary axis.*

In the following we use the notation

$$\mathcal{E} = \{v \in X^\gamma : Av = F(v)\},$$

$$\mathcal{E}^h = \{V \in \mathcal{V} : A^h V = F^h(V)\}$$

to denote the equilibria of (2.1) and of (2.6) or (2.7).

In [21] the lower semicontinuity of attractors is considered when (2.1) is in gradient form: it is supposed that a Lyapunov function $V(\bullet) : X^\gamma \mapsto \mathbb{R}$ exists for (2.1) in which case, provided the equilibria are hyperbolic, the attractor $\mathcal{A}$ is given by

$$\mathcal{A} = \overline{\bigcup_{v \in \mathcal{E}} W^u(v)}. \tag{4.7}$$

(See [20] for a precise definition of Lyapunov function in this context). Using the techniques of [21] it may then be shown, under assumptions I and II of Theorem 4.2, together with an assumption that the local unstable manifolds are lower-semicontinous in X^γ with respect to h that

$$\text{dist}_\gamma(\mathcal{A}, \mathcal{A}^h) \to 0 \;\; as \;\; h \to 0.$$

This is a lower semicontinuity result and the same method of proof has been employed in [28] to consider the effect of time discretization on lower-semicontinuity for ordinary differential equations in gradient form. The proof of [21] explicitly uses the Morse decomposition of the attractor induced by the gradient structure. However, in [29], [30] it is shown that the assumption that the system be in gradient form is *not required* to prove lower-semicontinuity with respect to certain perturbations. Rather it is sufficient to assume that the attractor has the form (4.7) where every element of $\mathcal{E}$ is hyperbolic. Here we generalise the theorem of [29], [30] which concerns ordinary differential equations, to equation (2.1).

Theorem 4.3 Lower-Semicontinuity of Attractors *Assume that:*

(I) equations (2.1), (2.6) *have global attractors* $\mathcal{A}$ *and* $\mathcal{A}^h$ *and, furthermore, that* $\mathcal{A}$ *is given by* (4.7) *where* $\mathcal{E}$ *comprises a finite number of hyperbolic equilibria;*

(II) for any $T, \delta, r > 0$ $\exists H_1 = H_1(T, \delta, r) > 0, C = C(T, r) > 0$ *such that*

$$|S(T)u - S^h(T)U|_\gamma \leq \delta + C|u - U|_\gamma,$$

for all $u \in \mathcal{B}_\gamma(0, r)$, $U \in \mathcal{B}_{h,\gamma}(0, r)$ *and* $h \in (0, H_1)$;

(III) for every $v \in \mathcal{E}$ *and* $\delta > 0$ $\exists r = r(v, \delta) > 0, H_2 = H_2(v, \delta) > 0$ *such that*

$$dist_\gamma(W^u(v, r), \mathcal{A}^h) \leq \delta$$

for all $h \in (0, H_2)$.

Then

$$dist_\gamma(\mathcal{A}, \mathcal{A}^h) \to 0 \ \ as \ \ h \to 0.$$

Proof. It is sufficient to prove that, given any $\epsilon > 0$ $\exists H > 0$ such that for every $y \in \mathcal{A}$, $\exists y^h \in \mathcal{A}^h$ with the property that $|y - y^h|_\gamma \leq 2\epsilon$ for $h \in (0, H)$. Since $\mathcal{A}$ is given by (4.7) we need only consider $y \in \overline{W^u(v)}$ for all $v \in \mathcal{E}$. Given $v \in \mathcal{E}$ we let r be given by assumption III.

Let $\partial\mathcal{B}_\gamma(v; r)$ denote the boundary of $\mathcal{B}_\gamma(v; r)$; that is

$$\partial\mathcal{B}_\gamma(v, r) = \{u \in X^\gamma : |u - v| = r\}.$$

Now set

$$\mathcal{Q} = \overline{W^u(v; r)} \bigcap \partial\mathcal{B}_\gamma(v; r)$$

and

$$\mathcal{W} = W^u(v) \backslash \overline{W^u(v; r)}.$$

Then, for r sufficiently small,

$$\mathcal{W} = \bigcup_{t>0} S(t)\mathcal{Q}.$$

Note that for every $\tau > 0$ and every point $u \in \mathcal{Q}$ there exists a $u^- \in W^u(v; r)$ such that $u = S(\tau)u^-$. Using this fact, together with the fact that $\mathcal{W}$ is a union of trajectories all of which start on $\mathcal{Q}$, the relative compactness of (2.2) shows that $\overline{\mathcal{W}}$ is compact in X^γ. Note that $\{\mathcal{B}_\gamma(x, \epsilon) \colon x \in \mathcal{W}\}$ is an ϵ-cover for $\overline{\mathcal{W}}$ in X^γ and hence, since $\overline{\mathcal{W}}$ is compact in X^γ, we may extract a finite sub-cover. Denote this subcover by $\{B_i(\epsilon)\}_{i=1}^I$ and note that each $B_i(\epsilon)$ contains a point $y_i \in \mathcal{W}$, where $B_i(\epsilon) = \mathcal{B}(y_i, \epsilon)$. By construction $\exists x_i \in \mathcal{Q}$ and

$T_i > 0$ such that $S(T_i)x_i = y_i$ for each $y_i \in \mathcal{W}$. Now assumption III implies that, for any $\delta > 0, \exists x_i^h \in \mathcal{A}^h$ and $H_2^i = H_2^i(v,\delta) > 0$ such that

$$|x_i - x_i^h|_\gamma \leq \delta \quad \forall h \in (0, H_2^i],$$

and by the invariance of $\mathcal{A}^h$ it follows that $y_i^h = S^h(t)x_i^h \in \mathcal{A}^h$. It now follows from assumption II that there exists $H_1^i = H_1^i(v,\delta) > 0$ such that

$$|y_i - y_i^h|_\gamma = |S(T_i)x_i - S^h(T_i)x_i^h| \leq (1+C)\delta \quad \forall h \in (0, \min\{H_1^i, H_2^i\}].$$

Since I is finite, we deduce that $\exists \{y_i^h\}_{i=1}^I$ and $H_3 = H_3(v,\epsilon)$ such that

$$\max_{1\leq i\leq I} |y_i - y_i^h|_\gamma \leq \epsilon \quad \forall h \in (0, H_3].$$

Thus, for every $y \in B_i(\epsilon)$ and $i : 1 \leq i \leq I$ there exists $y_i^h \in \mathcal{A}^h$ such that

$$|y - y_i^h|_\gamma \leq 2\epsilon \quad \forall h \in (0, H_3].$$

Since the $B_i(\epsilon), i = 1, \ldots, I$ form a cover of $\overline{\mathcal{W}}$ we deduce that

$$\text{dist}_\gamma\{\overline{\mathcal{W}}, \mathcal{A}^h\} \leq 2\epsilon \quad \forall v \in (0, H_3].$$

Noting that there are only a finite number of v defining the attractor through (4.7) and using assumption III we deduce that $\exists H = H(\epsilon)$:

$$\text{dist}_\gamma\{\mathcal{A}, \mathcal{A}^h\} \leq 2\epsilon \quad \forall h \in (0, H]$$

and the result follows. □

Remarks

(i) to establish III it is sufficient to show that, for every $v \in \mathcal{E}$ $\exists r = r(v,\delta) > 0$ and $H_2 = H_2(v,\delta) > 0$ and $V \in \mathcal{E}^h, r' > 0$ such that

$$\text{dist}_\gamma(W^u(v,r), W_h^u(V,r')) \leq \delta \tag{4.8}$$

for all $h \in (0, H)$. This is since the unstable manifold of a fixed point is necessarily contained in the global attractor.

(ii) if assumption III is replaced by (4.8) then the method of proof for Theorem 4.3 shows that the closure of the unstable manifold of an equilibrium point is lower-semicontinuous with respect to numerical perturbations.

(iii) it is not difficult to generalise Theorem 4.3 to include the effect of time-discretization. □

This concludes our analysis of the upper and lower semicontinuity of attractors with respect to numerical perturbations. Two important points have been established. Firstly, lower semicontinuity will not hold in general and thus it is natural to consider the effect of numerical approximation on more robust objects; this motivates the study of inertial manifolds which contain the global attractor and perturb smoothly. Secondly, the cases in which lower semicontinuity can be established all require proving the lower semicontinuity of local unstable manifolds of hyperbolic equilibria. Thus, in the final section we consider inertial and unstable manifolds under perturbation.

5 Attractive Invariant Manifolds

Motivated by the remarks at the end of the previous section, we now study the effect of numerical approximation on attractive invariant manifolds. We start by showing how the problems of the existence of both inertial manifolds and unstable manifolds can be formulated in the same framework – a framework which requires the study of globally Lipschitz mappings in a Banach space. We then state and sketch the proof of an abstract theorem concerning perturbation of attractive invariant manifolds for such globally Lipschitz mappings.

Inertial Manifolds.

Consider (2.1) in the case where the operator A is self-adjoint. For our discussion of inertial manifolds we assume that there exists $\beta \in (0,1)$ such that the operator F in (2.1) satisfies the following conditions: $\exists \gamma \geq 0, \beta \in [0,1)$ and $E(\sigma) > 0$ such that $F : X^\gamma \to X^{\gamma-\beta}$ satisfies

$$
\begin{aligned}
|F(u)|_{\gamma-\beta} &\leq E(\sigma) \quad \forall u \in \mathcal{B}_\gamma(0,\sigma) \\
|F(u) - F(v)|_{\gamma-\beta} &\leq E(\sigma)|u-v|_\gamma \quad \forall u, v \in \mathcal{B}_\gamma(0,\sigma);
\end{aligned}
\tag{5.1}
$$

We introduce the following decomposition of X :

$$
X = Y \oplus Z, \quad Y = span\{\psi_1, \psi_2, \ldots, \psi_q\}, \quad Z = Y^\perp
$$

where $A\varphi_i = \lambda_i \varphi_i$ (as in section 2) and the orthogonal complement is taken in X. We also denote by P and Q the projections $P : X \mapsto Y$ and $Q : X \mapsto Z$.

Now assume that (2.1) has an absorbing set $\mathcal{B}_\gamma(0,\rho)$. Then, if we are interested only in the long-time behaviour of (2.1), it is sufficient to modify F outside $\mathcal{B}_\gamma(0,\rho)$ in a smooth fashion to obtain a globally bounded and Lipschitz function. Doing this in the standard fashion (see [17]) we see that the following holds: $\exists \gamma \geq 0, \beta \in [0,1)$ and $E > 0$ such that $F : X^\gamma \to X^{\gamma-\beta}$ satisfies

$$
\begin{aligned}
|F(u)|_{\gamma-\beta} &\leq E \quad \forall u \in X^\gamma \\
|F(u) - F(v)|_{\gamma-\beta} &\leq E|u-v|_\gamma \quad \forall u, v \in X^\gamma
\end{aligned}
\tag{5.2}
$$

Thus we consider equation (2.1) under (5.2). Again a semigroup $S(t) : X^\gamma \mapsto X^\gamma$ may be defined so that $u(t) = S(t)u_0$.

An inertial manifold for (2.1) under (5.2) is a set $\mathcal{M}$ defined by a graph $\Phi \in C(Y,Z)$ so that

$$
\mathcal{M} = \{u \in X^\gamma : Qu = \Phi(Pu)\}
$$

satisfies

$$
u(0) \in \mathcal{M} \Rightarrow u(t) \in \mathcal{M} \; \forall t \in \mathbb{R}
$$

and, furthermore,

$$
\exists C, \mu > 0 : \; \text{dist}_\gamma(S(t)u_0, \mathcal{M}) \leq Ce^{-\mu t}\text{dist}_\gamma(u_0, \mathcal{M}) \quad \forall t \geq 0, \; u_0 \in X^\gamma.
$$

The existence of inertial manifolds was first considered in [17] using an approach based on the Lyapunov-Perron method familiar from the construction of center manifolds. Related results concerning the existence of inertial manifolds can be found in, for example, [38] [7], [8], [14]. Here we outline a different proof for the existence of inertial manifolds which is particularly convenient for the consideration of numerical perturbations; see [32] for details.

Let $U_m = S(mT)u_0$. Then use of the variation of constants formula in (2.1) shows that

$$U_{m+1} = G(U_m) \tag{5.3}$$

where $G : X^\gamma \mapsto X^\gamma$ satisfies

$$G(u) = Lu + N(u), \quad L := e^{-AT}, \quad N(u) = \int_0^T e^{-A(T-s)} F(S(s)u)ds. \tag{5.4}$$

Furthermore, using the spectral properties of A on Y and Z, together with (5.2) and the smoothing properties of A, shows that there exist positive constants a, b, c, B such that **Assumptions G** hold:

$$|Lz|_\gamma \leq a|z|_\gamma \qquad \forall z \in Z; \tag{G1}$$

$$\exists! w \in Y : Lw = p, \ \forall p \in Y \ \ \& \ \ b|y|_\gamma \leq |Ly|_\gamma \leq c|y|_\gamma \qquad \forall y \in Y; \tag{G2}$$

$$|\mathcal{R}(N(u) - N(v))|_\gamma \leq B|u - v|_\gamma \ \ \forall u, v \in X^\gamma, \quad |\mathcal{R}N(u)|_\gamma \leq B \ \ \forall u \in X^\gamma, \tag{G3}$$

where $\mathcal{R}$ equals either I, P or Q.

In particular we have

$$a = e^{-\lambda_{q+1}T}, \ b = e^{-\lambda_q T}, \ c = e^{-\lambda_1 T}, \ B \propto T^{1-\beta}. \tag{5.5}$$

If we let $p_m = Pu_m$ and $q_m = Qu_m$ then the graph Φ representing the inertial manifold must satisfy

$$p = L\xi + PN(\xi + \Phi(\xi)) \tag{5.6}$$

$$\Phi(p) = L\Phi(\xi) + QN(\xi + \Phi(\xi)) \tag{5.7}$$

together with the attractivity condition

$$|q_m - \Phi(p_m)|_\gamma \leq Ce^{-\mu m T}|q_0 - \Phi(p_0)|_\gamma. \tag{5.8}$$

To construct Φ we seek a fixed point of the mapping $\mathcal{T} : C(Y, Z) \mapsto C(Y, Z)$ defined by

$$p = L\xi + PN(\xi + \Phi(\xi)) \tag{5.9}$$

$$(\mathcal{T}\Phi)(p) = L\Phi(\xi) + QN(\xi + \Phi(\xi)). \tag{5.10}$$

Any $\Phi \in C(Y, Z)$ satisfying (5.6), (5.7), (5.8) is only an inertial manifold for the time T flow of the equation. It is necessary to show *a posteriori* that it is also invariant for the equation (2.1) for every $t > 0$; see [32] for details.

This completes our set-up of the problem of finding inertial manifolds as the graph of a function relating projections of the solution of a globally Lipsctiz mapping in a Banach space. After showing that the problem of existence for unstable manifolds may be set in the same framework, we consider the existence of Φ satisfying (5.6), (5.7) and (5.8) together with the effect of numerical perturbation. □

Unstable Manifolds.

Consider the equation

$$v_t + \mathcal{C}v = g(v), \quad v(0) = v_0 \tag{5.11}$$

where $\mathcal{C}$ is a densely defined, positive-definite, sectorial operator on X. Thus we may define the Banach spaces $\mathcal{X}^\alpha := D(\mathcal{C}^\alpha)$ with norm $\|u\|_\alpha := |\mathcal{C}^\alpha u|, \alpha \in [0, \infty)$. We assume further that $g : \mathcal{X}^\gamma \mapsto X$ is locally Lipschitz for some $\gamma \in [0, 1)$. Let (5.11) have an equilibrium point $\bar{v} \in \mathcal{X}^\gamma$ satisfying

$$\mathcal{C}\bar{v} = g(\bar{v}). \tag{5.12}$$

Introducing $u = v - \bar{v}$ we may write (5.11) as

$$u_t + Au = F(u), \quad u(0) = u_0 := v_0 - \bar{v} \tag{5.13}$$

where

$$\begin{aligned} A &:= \mathcal{C} - dg(\bar{v}), \\ F(u) &:= g(\bar{v} + u) - g(\bar{v}) - dg(\bar{v})u, \end{aligned} \tag{5.14}$$

We assume that the operator A has spectrum $\{\lambda_i\}_{i=1}^\infty$ satisfying

$$Re(\lambda_1) \leq \ldots \leq Re(\lambda_q) < 0 < Re(\lambda_{q+1}) \leq \ldots$$

for some integer $q \geq 1$. This ensures that $\bar{v}$ is an unstable equilibrium point of (5.11) so that the unstable manifold is non-trivial.

We assume that $dg(\bar{v})$ is a bounded linear map from $\mathcal{X}^\gamma$ to X so that $|dg(\bar{v})\mathcal{C}^{-\gamma}|$ is bounded; this implies that A is sectorial – see Corollary 1.4.5 of [23]. Thus we may construct from A the Banach spaces X^γ as in section 2. From this it follows, by application of Theorem 1.4.8 in [23], that the Banach spaces $\mathcal{X}^\alpha$ and X^α are equivalent and that $\| \bullet \|_\alpha \approx | \bullet |_\alpha$.

Finally, we assume that there exists $E(\rho)$ with $E(\rho) \to 0$ as $\rho \to 0_+$ such that, for all $u_1, u_2 \in \mathcal{B}_\gamma(0, \rho)$

$$|F(u_1) - F(u_2)| \leq E(\rho)|u_1 - u_2|_\gamma.$$

This guarantees the existence of a local solution $u(t) \in X^\gamma$ to (5.13) in the neighbourhood of $u = 0$ (equivalently for (5.11) in a neighbourhood of $v = \bar{v}$). As in the inertial manifold case we can modify the function g outside a small neighbourhood of $\bar{v}$, and hence F outside a small neighbourhood of 0, in such a way that (5.13) now satisfies the condition that there exists $E(\rho)$ with $E(\rho) \to 0$ as $\rho \to 0_+$ such that, for all $u_1, u_2 \in \mathcal{X}^\gamma$

$$|F(u_1) - F(u_2)| \leq E(\rho)|u_1 - u_2|_\gamma.$$

As for the construction of inertial manifolds we introduce a splitting of the space $X = Y \oplus Z$, where now Y and Z are found by spectral projections associated with the spectral sets $Re(\lambda) \leq Re(\lambda_q)$ and $Re(\lambda) \geq Re(\lambda_{q+1})$ respectively. As before $P : X \mapsto Y$ and $Q : X \mapsto Z$. By a similar process of using the variation of constants formula, we obtain (5.3), (5.4) where, once again, Assumptions G are satisfied. In this case, by appropriate choice of T, it may be shown that

$$a = \frac{1}{2}, \quad b = 2, \quad c \geq 2, \quad B = B(\rho) \to 0 \; as \; \rho \to 0_+. \tag{5.15}$$

Again the unstable manifold may be found as a graph $\Phi \in C(Y, Z)$ satisfying (5.6), (5.7). Thus it is possible to seek such a Φ as a fixed point of the mapping $\mathcal{T}$ defined by (5.9), (5.10).

This completes our set-up of the problem of finding unstable manifolds as the graph of a function relating projections of the solution of a globally Lipschitz mapping in a Banach space. □

A numerical approximation of (2.1) or (5.11) will yield a mapping

$$U^h_{m+1} = G^h(U^h_m) \tag{5.16}$$

where $G^h : X^{h,\gamma} \mapsto X^{h,\gamma}$ is defined by

$$G^h(u) := L^h u + N^h(u). \tag{5.17}$$

We will not be more specific about the definition of L^h and N^h since this depends upon whether inertial manifolds or unstable manifolds are being considered. We simply assume that the space $\mathcal{V}$ may be decomposed as $\mathcal{V} : Y^h \oplus Z^h$ and introduce the projections $P^h : \mathcal{V} \mapsto Y^h$, $\quad Q^h : \mathcal{V} \mapsto Z^h$. Here Y^h and Z^h may be considered as approximations to Y and Z. We denote by E^h the $X-$projection $E^h : X \mapsto \mathcal{V}$.

For a wide variety of numerical methods it is possible to show that the following **Assumptions** G^h are satisfied: there exist positive constants a, b, c, B, C and $C(\rho)$ such that:

$$|L^h z|_{h,\gamma} \leq a|z|_{h,\gamma} \qquad \forall z \in Z^h; \tag{G^h1}$$

$$\exists! w^h \in Y^h : L^h w^h = q^h, \forall q^h \in Y^h \;\; \& \;\; b|y|_{h,\gamma} \leq |L^h y|_{h,\gamma} \leq c|y|_{h,\gamma} \quad \forall y \in Y^h; \tag{G^h2}$$

$$|\mathcal{R}(N^h(u) - N^h(v))|_{h,\gamma} \leq B|u - v|_{h,\gamma} \;\; \forall u, v \in V^h \quad |\mathcal{R}(N^h(u))|_{h,\gamma} \leq B \;\; \forall u \in \mathcal{V} \tag{G^h3}$$

where $\mathcal{R}$ equals either I, P^h or Q^h;

$$|P - P^h|_\gamma \leq Ch; \tag{G^h4}$$

$$|G(u) - G^h(u^h)|_\gamma \leq C(\rho)(h + |u - u^h|_\gamma) \; \forall u \in \mathcal{B}_\gamma(0,\rho), \; u^h \in \mathcal{B}_\gamma(0,\rho) \bigcap \mathcal{V}; \tag{G^h5}$$

$$|E^h|_\gamma, |P|_\gamma, |P^h|_\gamma \leq C; \tag{G^h6}$$

$$C^{-1}|u|_\gamma \leq |u|_{h,\gamma} \leq C|u|_\gamma \quad \forall u \in \mathcal{V}. \tag{G^h7}$$

The choice of the constants a, b, c and B as being the same as those in Assumptions G may be acheived without loss of generality. Assumptions $(G^h1) - (G^h3)$ are analogous to Assumptions $(G1) - (G3)$ whilst Assumptions $(G^h4) - (G^h7)$ concern the relationship between the approximate mapping and the original mapping.

As for the continuous case, an attractive invariant manifold can be represented as a graph $\Phi^h : C(Y^h, Z^h)$ satisfying

$$\begin{aligned} p &= L^h\xi + PN^h(\xi + \Phi^h(\xi)) \quad (5.18) \\ \Phi^h(p) &= L^h\Phi^h(\xi) + QN^h(\xi + \Phi^h(\xi)) \quad (5.19) \end{aligned}$$

together with the attractivity condition

$$|q^h_m - \Phi(p^h_m)|_\gamma \leq Ce^{-\mu m T}|q^h_0 - \Phi(p^h_0)|_\gamma. \quad (5.20)$$

Here $p_m = Pu_m$ and $q_m = Qu_m$.

The following theorem shows that provided certain conditions on a, b, c and B are satisfied (those yielding existence of attractive invariant manifolds for the mappings (5.3) and (5.16)), then upper and lower semicontinuity of attractive invariant manifolds may be shown under Assumptions G and G^h.

Theorem 5.1 Continuity of Invariant Manifolds *Assume that the mappings* (5.3) *and* (5.16) *satisfy Assumptions G and Assumptions G^h respectively. Assume further that there exist constants $\delta', \epsilon' \in (0,\infty)$, $\mu \in (0,1)$ and $K \in (1,\infty)$ such that:*

$$b^{-1}B(1+\delta) \leq \mu, \quad (C1)$$

$$a\epsilon + B \leq \epsilon, \quad (C2)$$

$$\theta := a\delta + B(1+\delta) \leq \delta\phi, \quad (C3)$$

where $\phi := b - B(1+\delta) > 0$ by (C1) and

$$a + B(1+\delta) \leq \mu, \quad (C4)$$

for all $\delta \in (\delta', K\delta')$ and $\epsilon \in (\epsilon', K\epsilon')$. Then the mappings (5.3), (5.16) *both possess attractive invariant manifolds representable as graphs $\Phi : Y \mapsto Z$ and $\Phi^h : Y^h \mapsto Z^h$ respectively and satisfying* (5.6), (5.7), (5.8) *and* (5.18), (5.19), (5.20) *respectively. Furthermore if* either

$$c < 1 \;\&\; \exists r > 0 : |N(u)|_\gamma = 0 \qquad \forall u : |Pu|_\gamma \geq r \quad (5.21)$$

or

$$b > 1 \quad (5.22)$$

then:

(i) for any $p \in Y$ there exists $C(p) > 0$ such that

$$|(p + \Phi(p)) - (P^h p + \Phi^h(P^h p))|_\gamma \leq C(p)h;$$

(ii) for any $p^h \in Y^h$ there exists $C(p^h) > 0$ such that

$$|(Pp^h + \Phi(Pp^h)) - (p^h + \Phi^h(p^h))|_\gamma \leq C(p^h)h.$$

Sketch Proof The details of the proof can be found in [32]. The basic idea to establish existence is to use the contraction mapping theorem. Conditions (C1)–(C3) enable this for the mappings (5.3) and (5.16) under Assumptions $(G1)-(G3)$ and $(G^h1)-(G^h3)$ respectively. Condition (C4) yields the required exponential attraction. To prove the convergence result relating the true and approximate manifolds uses a modification of the uniform contraction principle, using (G^h5) to give the required continuity with respect to perturbations. However, since Φ and Φ^h are defined as graphs over different spaces this application of the uniform contraction principle is not entirely straightforward. The assumptions $(G^h4), (G^h6)$ and (G^h7) are used to get around this difficulty. □

Remarks

(i) The conditions (C1)–(C4) can be satisfied in the inertial manifold case provided that both λ_q and $\lambda_{q+1} - \lambda_q$ can be made sufficiently large. This is known as the *spectral gap condition* in [17] and identical conditions are derived in the existence proof sketched here; see [32]. For unstable manifolds (C1)–(C4) are satisfied since $a < 1 < b$ and B can be made arbitrarily small by choice of ρ.

(ii) The method of proof of the continuity result is a generalisation of that used by [5] to prove convergence of center-unstable manifolds in ordinary differential equations under numerical approximation. In the context of unstable manifolds for partial differential equations, a very similar *existence proof* can be found in [3].

(iii) The first proof concerning convergence of unstable manifolds under numerical approximation may be found in [4] where ordinary differential equations are considered. In [1] the effect of time-discretization on the unstable manifold of scalar reaction-diffusion equations is studied whilst in [36] the same question is considered under finite element spatial approximation. See also [21].

(iv) The convergence of inertial manifolds under spectral approximation based on the eigenfunctions of A is studied in [17], [18]. The same problem is considered for a specific time discretization in [9].

(v) The abstract framework described here used to study the existence and convergence of inertial manifolds in semi and fully discrete finite element approximations of a scalar reaction-diffusion equation and the Cahn-Hilliard equation in [32]. Furthermore, the existence and convergence of local unstable manifolds in a fully discrete reaction-diffusion equation is also analysed.

(vi) It is worth noting the different methods and assumptions employed in constructing invariant manifolds in approximation schemes. The papers by [9] and [36] are based on the Lyapunov-Perron type existence theory and, as such, require the derivation of non-standard error bounds over long-time intervals together with certain spectral approximation properties. The paper [1] is also based on a Lyapunov-Perron type approach and requires a C^1 approximation result over finite time intervals. The result of [32] described here is based on the Hadamard graph transform and requires standard C^0 error bounds on finite time intervals (G^h5) together with the closeness of certain spectral sets and their associated projections (G^h4).

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