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Attractive Invariant Manifolds under Approximation. Inertial Manifolds

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A class of nonlinear dissipative partial differential equations that possess finite dimensional attractive invariant manifolds is considered. An existence and perturbation theory is developed which unifies the cases of unstable manifolds and inertial manifolds into a single framework. It is shown that certain approximations of these equations, such as those arising from spectral or finite element methods in space, one-step time-discretization or a combination of both, also have attractive invariant manifolds. Convergence of the approximate manifolds to the true manifolds is established as the approximation is refined. In this part of the paper applications to the behavior of inertial manifolds under approximation are considered. From this analysis deductions about the structure of the attractor and the flow on the attractor under discretization can be made. © 1995 Academic Press, Inc.

1. INTRODUCTION

Attractive invariant manifolds for evolution equations are fundamental in understanding long-time dynamics. Important examples of attractive invariant manifolds that we consider are the unstable manifolds of hyperbolic equilibria and inertial manifolds. The purpose of this work is to study the behavior of such manifolds under perturbations sufficiently general to include the effect of numerical approximation. The abstract evolution equation

$$\frac{du}{dt} + Au = F(u), \qquad u(0) = u_0. \tag{1.1}$$

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is considered. In this paper we describe a general existence and perturbation theory for attractive invariant manifolds which encapsulates both inertial manifolds and unstable manifolds in one framework; the perturbation results for these manifolds are appropriate for the analysis of attractive invariant manifolds under discretization. The general approximation theory is formulated using semigroups; using this theory it is shown that standard approximations of the equation (1.1), encompassing finite element, spectral and time discretization, have an approximate attractive invariant manifold which converges to the true invariant manifold as the approximation is refined. We also describe a number of applications of the general theory; in particular, we prove existence and convergence of inertial and unstable manifolds in a variety of partial differential equations under discretization. However, the perturbation theory is not restricted to numerical approximation schemes and applications to a singularly perturbed partial differential equation are described in [19].

There are several reasons why it is important to understand the behavior of the attractive invariant manifolds of (1.1) under numerical approximations. It is well known that standard error estimates for *individual trajectories* are useless over long-time intervals since they typically contain a constant which grows exponentially with the time interval; thus in recent years there has been considerable effort to understand the behavior of invariant objects of a dynamical systems such as (1.1) under discretization–see, for examples, [3, 5, 10, 15, 21, 27, 29, 30, 32, 35, 36, 43–48, 56, 57, 61, 62, 64]. In particular, the work of [29] shows that the global attractor for (1.1) is upper semicontinuous under approximation; since the attractor may not attract exponentially, it is not possible to establish lower semicontinuity except under very special circumstances–see [30, 34, 36, 43, 46]. The first of these papers assumes that the system is in gradient form and the remainder the slightly weaker assumption that the attractor is the union of unstable manifolds of equilibria. The papers [36, 43] both contain simple counter examples showing why lower semicontinuity does not hold in general.

A major motivation for studying the inertial manifold under numerical approximation is that it is an exponentially attracting invariant manifold (which contains the global attractor) and so there is reason to suspect that it is both upper and lower semicontinuous under numerical approximation; such a result is proved in part I of this paper. This then shows that the inertial form for (1.1), that is, the PDE restricted to its inertial manifold, and its approximation are close. This is a step towards establishing a relationship between the dynamics of (1.1) and its approximation. More precisely if the inertial forms are C^1 close, then one can apply a result of [53] to make deductions about the relationships between the true and approximate flows on the attractor (see [41, 42]).

A major motivation for studying the unstable manifold under numerical approximation is the following: the work of [30, 34] shows that, when the global attractor is the closure of the union of unstable manifolds of hyperbolic equilibrium points, it is both upper and lower semicontinuous. To prove this result requires an error bound for the approximation of local unstable manifolds; in part II of this paper we show how the theory developed for inertial manifolds can be modified to incorporate the case of unstable manifolds.

We consider (1.1) as an ordinary differential equation in a separable Hilbert space X with inner product $(\bullet, \bullet)$ and norm $|\bullet|^2 = (\bullet, \bullet)$. We assume that A is a densely defined sectorial operator with compact inverse, eigenvalues $\{\lambda_i\}$ and associated eigenfunctions $\{\varphi_i\}$. Thus it is possible to choose $\zeta \geqslant 0$ such that all eigenvalues of

$$\tilde{A} := A + \zeta I$$

have strictly positive real part. Hence we may set $X^\alpha = D(\tilde{A}^\alpha)$ where $\tilde{A}^\alpha = (\tilde{A}^{-\alpha})^{-1}$ and for $0 < \alpha < 1$

$$\tilde{A}^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\tilde{A}t}\, dt; \tag{1.2}$$

see [52]. For $\alpha = 0$ we define $\tilde{A}^0 = I$. Then X^α is a Hilbert space with norm $|\bullet|_\alpha^2 = (\tilde{A}^\alpha \bullet, \tilde{A}^\alpha \bullet)$. The operator A generates an analytic semigroup $L(t)$. We assume that F satisfies sufficient conditions so that (1.1) generates a semigroup $S(t)\colon X^\gamma \mapsto X^\gamma$, for some $\gamma \geqslant 0$, which is Lipschitz continuous for each $t \geqslant 0$. We employ the notation

$$\mathscr{B}_\alpha(0, r) = \{u \in X^\alpha \colon |u|_\alpha \leqslant r\}.$$

Since our aim is to study Eq. (1.1) under approximation and since standard error estimates for numerical schemes are formulated over finite time intervals, we find it convenient to write the solutions of Eq. (1.1) as solutions of the time T map of the flow. The solution of (1.1) at time t can be written

$$u(t) = L(t)\, u_0 + N(u_0, t), \tag{1.3}$$

where

$$L(t) := e^{-At}, \qquad N(v, t) = \int_0^t L(t-s)\, F(S(s)\, v)\, ds. \tag{1.4}$$

The sequence $u_m := u(mT)$ then satisfies the map

$$u_{m+1} = G(u_m) \tag{1.5}$$

where

$$G(u) := Lu + N(u), \qquad L := L(T), \qquad N(v) := N(v, T). \tag{1.6}$$

We formulate an existence theory for attractive invariant manifolds of the mapping (1.5), such as inertial manifolds and unstable manifolds, which is based on the contraction mapping theorem; the technique is similar to that used in [2] for the construction of unstable manifolds (see also [63]) and to that used in [50] where singularly perturbed ODEs are studied. It is based on the Hadamard graph transform and as such is related to the method used in [49] to construct the inertial manifold; however, our proof differs from the proof of [49] in that explicit use of a cone condition and squeezing property is not needed. This does not yield any new results but leads to a simple exposition which, furthermore, is directly amenable to a perturbation theory exploiting known finite-time error estimates for approximation schemes. We note that there are already several results concerning the existence and convergence of inertial manifolds in approximate schemes (see [25, 26] for Galerkin approximation and [12] for a specific time discretization) and unstable manifolds in approximation schemes (see [1, 4, 48]); however, these methods of proof are based on the Lyapunov-Perron existence theory and are not amenable to the direct use of standard finite time error bounds.

Our goal is to provide a single framework in which error bounds for the approximation of attractive invariant manifolds can be obtained by application of existing (or easily derived) finite time error estimates and spectral approximation theory. Of course, no such framework can be all encompassing in such a broad setting but we believe that the approach described here includes many situations and that, furthermore, it provides a methodology that can be readily adapted to other different situations. The approach we take is similar to that developed in [5] where the behavior of center-unstable manifolds of ordinary differential equations under approximation is studied.

To construct an attractive invariant manifold we decompose the space X in the usual way: let P denote the spectral projection associated with $\{\varphi_1, ..., \varphi_q\}$, the first q eigenfunctions of A, and $Q = I - P$. We do not assume that P is self-adjoint; however, $Y = PX$ and $Z = QX$ do provide a decomposition of the space $X = Y \oplus Z$. Under the conditions on A and F described above we may define an operator G: $X^\gamma \mapsto X^\gamma$ by (1.6). We make the following **Assumptions** G on the mapping G. We suppose that, given an integer q for which the projections P and Q are defined, there exist positive constants a, b, c, B such that:

$$|Lz|_\gamma \leqslant a\,|z|_\gamma \qquad \forall z \in Z; \tag{G1}$$

$$\exists! w \in Y\colon Lw = p, \qquad \forall p \in Y \quad \& \quad b\,|y|_\gamma \leqslant |Ly|_\gamma \leqslant c\,|y|_\gamma \qquad \forall y \in Y; \tag{G2}$$

$$|\mathscr{R}(N(u) - N(v))|_\gamma \leqslant B\,|u - v|_\gamma \qquad \forall u, v \in X^\gamma, \qquad |\mathscr{R}N(u)|_\gamma \leqslant B \qquad \forall u \in X^\gamma, \tag{G3}$$

where $\mathscr{R}$ equals either I, P or Q.

Assumptions $(G1)$ and $(G2)$ concern the spectrum of A whilst assumptions $(G3)$ are global bounds arising from the nonlinearity of the problem. Such global bounds are usually obtained by exploiting a suitable cut-off function for (1.1) in X^γ and using a *prepared equation* as in [6, 25]. See Section 4.

Our main interest in Eq. (1.5) is the existence of an inertial manifold or an unstable manifold which, after employing the cut-off function, is globally represented as a graph $\Phi\colon Y \mapsto Z$. We will seek such a manifold which is invariant under G and hence satisfies:

$$Qu_m = \Phi(Pu_m) \Leftrightarrow Qu_{m+1} = \Phi(Pu_{m+1}). \tag{1.7}$$

In the next section we give specific conditions on the constants appearing in Assumptions G which guarantee the existence of a finite-dimensional invariant manifold for Eq. (1.5). Moreover, the manifold defined by $\mathscr{M} = Graph(\Phi)$ will be shown to be attractive in the sense that there exist $\mu \in (0, 1)$ and $c_1 \in (0, \infty)$:

$$dist_{X^\gamma}(u_m, \mathscr{M}) \leqslant c_1 \mu^m, \tag{1.8}$$

for all $u_m \in X^\gamma$. In the inertial manifold case the assumptions made on the problem are essentially the same as those derived in [8, 9, 23, 25, 26, 49, 54, 55], namely, the spectrum of the operator A is required to satisfy a spectral gap condition.

Notice also in the inertial manifold case that the reduction of the map G to the inertial manifold yields the finite-dimensional map, called an inertial form,

$$p_{m+1} = PG(p_m + \Phi(p_m)). \tag{1.9}$$

Moreover, since the global attractor is compact and the inertial manifold attracts all trajectories, the inertial manifold contains the attractor, and hence, the long-time dynamics of the inertial form is the same as that of the original (possibly) infinite-dimensional map G. This property is the main motivation behind the study of inertial manifolds.

We will assume that the the approximation of (1.1) yields a sectorial operator A^h, approximating A, and with eigenvalues $\{\lambda_m^h\}$ and eigenfunctions $\{\varphi_m^h\}$. As in the continuous case $\exists \zeta > 0$ such that all eigenvalues of

$$\tilde{A}^h := A^h + \zeta I$$

have positive real part. We denote by u_m^h the approximation to u_m which lies in some space $V^h \subset X^\gamma$.

We let E^h: $X \mapsto V^h$ denote the projection onto the approximating subspace and denote by P^h: $X \mapsto Y^h$ the projection onto the first m eigenfunctions of A^h and Q^h: $X \mapsto Z^h$ by $Q^h = E^h - P^h$ where

$$V^h = Y^h \oplus Z^h, \quad Y^h = P^h X, \quad Z^h = Q^h X.$$

Note that

$$P^h = E^h P^h = P^h E^h, \qquad Q^h = E^h Q^h = Q^h E^h.$$

The operator A^h generates an analytic semigroup $L^h(t)$ on V^h and we can define the Hilbert spaces $X^{h,\alpha} = \mathscr{D}((\tilde{A}^h)^\alpha)$ with norm $|\bullet|_{h,\alpha}$ given by $|u|_{h,\alpha} = |(\tilde{A}^h)^\alpha u|$. The approximation process then yields a mapping

$$u_{m+1}^h = G^h(u_m^h). \tag{1.10}$$

where the operator G^h is assumed to take the form

$$G^h(u^h) = L^h u^h + N^h(u^h) \tag{1.11}$$

and L^h, N^h: $X^{h,\gamma} \mapsto X^{h,\gamma}$. We employ the notation

$$\mathscr{B}_{h,\gamma}(0, r) = \{u \in X^{h,\gamma} : |u|_{h,\gamma} \leqslant r\}.$$

We make the following **Assumptions** G^h concerning the mapping (1.10), (1.11). Given the same integer q as in Assumptions G, there exist positive constants a, b, c, B, C and $C(\rho)$ such that:

$$|L^h z|_{h,\gamma} \leqslant a\, |z|_{h,\gamma} \qquad \forall z \in Z^h; \tag{G^h1}$$

$$\exists! w^h \in Y^h : L^h w^h = q^h, \qquad \forall q^h \in Y^h \qquad \text{and} \qquad b\, |y|_{h,\gamma} \leqslant |L^h y|_{h,\gamma} \leqslant c\, |y|_{h,\gamma}$$
$$\forall y \in Y^h; \tag{G^h2}$$

$$|\mathscr{R}(N^h(u) - N^h(v))|_{h,\gamma} \leqslant B\, |u - v|_{h,\gamma} \qquad \forall u, v \in V^h \qquad |\mathscr{R}(N^h(u))|_{h,\gamma} \leqslant B$$
$$\forall u \in V^h \tag{G^h3}$$

where $\mathscr{R}$ equals either I, P^h or Q^h;

$$|P - P^h|_\gamma \leqslant Ch; \tag{G^h4}$$

$$|G(u) - G^h(u^h)|_\gamma \leqslant C(\rho)(h + |u - u^h|_\gamma)$$

$$\forall u \in \mathscr{B}_\gamma(0, \rho), \qquad u^h \in \mathscr{B}_\gamma(0, \rho) \cap V^h; \tag{G^h5}$$

$$|E^h|_\gamma, |P|_\gamma, |P^h|_\gamma \leqslant C; \tag{G^h6}$$

$$C^{-1} |u|_\gamma \leqslant |u|_{h,\gamma} \leqslant C |u|_\gamma \qquad \forall u \in V^h. \tag{G^h7}$$

Throughout the paper C, K will denote positive generic constants independent of h.

Assumptions (G^h1)–(G^h3) are simply discrete analogs of $(G1)$–$(G3)$ whilst assumptions (G^h4)–(G^h7) concern the approximation process and are readily established for a number of standard approximation schemes as we show in Sections 5, 6 and 7. It follows from these assumptions that an invariant manifold Φ^h: $Y^h \mapsto Z^h$ exists for the mapping (1.10) under the same conditions on the constants a, b, C, B that yield such a manifold $\mathscr{M}$ for (1.5). Furthermore $\mathscr{M}^h = Graph(\Phi^h)$ satisfies the invariance condition

$$Q^h u_m^h = \Phi^h(P^h u_m^h) \Leftrightarrow Q^h u_{m+1}^h = \Phi^h(P^h u_{m+1}^h). \tag{1.12}$$

and is exponentially attracting: $\exists c_1 > 0$ and $\mu \in (0, 1)$ such that

$$dist_{X^{h,\gamma}}(u_m^h, \mathscr{M}^h) \leqslant c_1 \mu^m. \tag{1.13}$$

As with the map G, the reduction of the map G^h to its inertial manifold gives the lower dimensional system

$$p_{m+1}^h = P^h G^h(p_m^h + \Phi^h(p_m^h)) \tag{1.14}$$

for $p^h \in Y^h$. Under the conditions we impose, the map (1.14) will have the same dimension as the map (1.9), the inertial form for the map G, uniformly as $h \to 0$. Furthermore, the inertial forms (1.9), (1.14) contain all the information about the long-time dynamics of the maps G, and G^h respectively. Since the inertial forms are finite dimensional, it would be desirable to use results such as those in [20, 53] concerning C^1 perturbations of finite-dimensional dynamical systems to study the infinite-dimensional problem. In particular, it is known that certain compact overflowing, inflowing, invariant (normally hyperbolic) manifolds persist under such perturbations (more generalstructures are considered in [53]). Stable stationary and periodic orbits are examples of such invariants sets. Thus we expect results like [10, 61, 62] (which are proven for the Navier–Stokes equations) to be a consequence of the C^1 closeness of the inertial forms for the equations under study here. The C^1 closeness is investigated in [41].

We remark that Eq. (1.14) can be thought of as an *approximate inertial form* as studied in, for example, [14, 24, 26, 38, 40, 59, 60] and references therein. In general, an approximate inertial form attempts to better approximate the long-time dynamics of the map G by enslaving the higher modes $Q^h u^h$ through a function such as Φ^h. The difference here is that one may approximate Φ^h exactly by working with the full map G^h, whereas in the above works one approximates either Φ or Φ^h directly and studies the lower dimensional approximate inertial form.

Our main theorem concerning the relationship between the attractive invariant manifolds of the true and approximate map is as follows:

MAIN THEOREM I. *Under Assumptions G, Assumptions G^h and Conditions C' (see Section 2) the mappings (1.5), (1.10) both possess attractive invariant manifolds representable as graphs Φ: $Y \mapsto Z$ and Φ^h: $Y^h \mapsto Z^h$ respectively and satisfying (1.7), (1.8) and (1.12), (1.13) respectively. Furthermore if* either

$$c < 1 \qquad \textit{and} \qquad \exists r > 0\colon |N(u)|_\gamma = 0 \qquad \forall u\colon |Pu|_\gamma \geqslant r \tag{1.15}$$

or

$$b > 1, \tag{1.16}$$

then:

(i) *for any $p \in Y$ there exists $C(p) > 0$ such that*

$$|(p + \Phi(p)) - (P^h p + \Phi^h(P^h p))|_\gamma \leqslant C(p)\, h;$$

(ii) *for any $p^h \in Y^h$ there exists $C(p^h) > 0$ such that*

$$|(Pp^h + \Phi(Pp^h)) - (p^h + \Phi^h(p^h))|_\gamma \leqslant C(p^h)\, h.$$

This shows the upper and lower semicontinuity of the attractive invariant manifolds with respect to the class of perturbations considered. The case where (1.15) holds is appropriate for the consideration of inertial manifolds where, typically, all eigenvalues of A are negative yielding $c < 1$ whilst the case (1.16) is appropriate for the consideration of unstable manifolds where the real parts of the eigenvalues of A on Y are strictly positive yielding $b > 1$.

The Main Theorem applies to nonlinear mappings in a Banach space with globally bounded and Lipschitz nonlinear part. To use this result for equations arising in applications it will be necessary to localize the invariant manifolds constructed. To this end we introduce the notation

$$\mathscr{M}_{loc,\gamma}(v, r) = \mathscr{M} \cap \mathscr{B}_{\gamma}(v, r), \tag{1.17}$$

$$\mathscr{M}^h_{loc,\gamma}(v, r) = \mathscr{M}^h \cap \mathscr{B}_{h,\gamma}(v, r). \tag{1.18}$$

We will also find it convenient to define the distance between a point and a set in X^γ via

$$dist_{X^\gamma}(x, B) = \inf_{y \in B} |x - y|_\gamma$$

and the semi-distiance between sets

$$dist_{X^\gamma}(B, C) = \sup_{x \in B} dist(x, C),$$

and analogously the quantity $\text{dist}_{X^{h,\gamma}}$.

In Section 2 we employ the contraction mapping theorem to prove the existence of an attractive invariant manifold for (1.5) under Assumptions G. Section 3 is concerned with proving the closeness of the manifolds $\mathscr{M}$ for (1.5) and the manifold $\mathscr{M}^h$ for (1.10). In Section 4 we show how the theory of Section 2 may be used to prove the existence of inertial manifolds for partial differential equations. Section 5 concerns the existence and semicontinuity of inertial manifolds for certain semi and fully discrete approximations of (1.1) based on spectral approximations of (1.1) based on the eigenfunctions of A. Section 6 concerns similar questions to Section 5 for finite element method approximations of a reaction-diffusion equation.

In part II of the paper unstable manifolds are studied. A basic existence theory is described, highlighting the parallels with the analogous theory for inertial manifolds in Section 5. Furthermore, the upper and lower semicontinuity of local unstable manifolds is proved. The upper and lower semicontinuity of the global unstable manifolds and of attractors are then studied using these results.

2. Existence Theory

In this section we prove the existence of an attractive invariant manifold for the mapping (1.5) (and hence for (1.10)). We will suppose throughout that **Conditions** C hold: given the constants a, b and B from Assumptions G and G^h, there exist constants $\delta, \varepsilon \in (0, \infty)$ and $\mu \in (0, 1)$ such that:

$$b^{-1}B(1+\delta) \leqslant \mu. \tag{C1}$$

$$a\varepsilon + B \leqslant \varepsilon. \tag{C2}$$

$$\theta := a\delta + B(1+\delta) \leqslant \delta\phi, \tag{C3}$$

where $\varphi := b - B(1+\delta) > 0$ by $(C1)$.

$$a + B(1+\delta) \leqslant \mu. \tag{C4}$$

In the case of inertial manifolds the satisfaction of $(C1)$–$(C2)$ corresponds to the operator A having sufficiently large eigenvalues whilst $(C3)$ corresponds to sufficiently large spectral gaps. The condition $(C4)$ follows under a combination of such conditions. For unstable manifolds $(C1)$–$(C4)$ are readily satisfied because $a < 1 < b$ and $B \ll 1$.

DEFINITION 2.1. Let $\Gamma = \Gamma(\delta, \varepsilon)$ denote the closed subset of $C(Y, Z)$ satisfying

$$\|\Psi\|_{\Gamma} := \sup_{p \in Y} |\Psi(p)|_{\gamma} \leqslant \varepsilon,$$

$$|\Psi(p_1) - \Psi(p_2)|_{\gamma} \leqslant \delta\, |p_1 - p_2|_{\gamma} \qquad \forall p_1, p_2 \in Y.$$

Furthermore $\Gamma^h(\delta, \varepsilon)$ is defined analogously with the norms taken in $X^{h, \gamma}$.

In order for the existence theorem for the attractive invariant manifold to hold for certain approximating systems, we will need Conditions C to hold for a range of parameters. That is, we may require:

Conditions C'. There exist $\delta', \varepsilon' > 0$ and $\mu \in (0, 1)$ such that given $K > 0$, Conditions $(C1)$–$(C4)$ hold for all $\delta \in [\delta', K\delta']$, $\varepsilon \in [\varepsilon', K\varepsilon']$.

THEOREM 2.2. *Suppose that Assumptions G and Conditions C' hold for the mapping G given by* (1.5), (1.6). *Then there exists a unique $\Phi \in \Gamma(\delta, \varepsilon)$ such that $\mathcal{M} = \mathrm{Graph}(\Phi)$ satisfies* (1.7) *and* (1.8). *The manifold $\mathrm{Graph}(\Phi)$ is independent of $\delta \in [\delta', K\delta']$, $\varepsilon \in [\varepsilon', K\varepsilon']$. Furthermore, if* (1.15) *holds and $R \geqslant cr + B$, then*

$$\Phi(p) \equiv 0 \ \ \forall p\colon |p|_{\gamma} \geqslant R. \tag{2.1}$$

Throughout the remainder of this section we assume that the conditions of Theorem 2.2 are satisfied without stating this explicitly in every result. The proof of the theorem will be given in a series of lemmas. With P, Q defined as in Section 1 we may write (1.5) as

$$p_{m+1} = Lp_m + PN(p_m + q_m) \tag{2.2}$$

$$q_{m+1} = Lq_m + QN(p_m + q_m) \tag{2.3}$$

for $u_m = p_m + q_m$ with $p_m \in Y$, $q_m \in Z$ and $m \geqslant 0$. An invariant manifold Φ is a fixed point of the operator T: $C(Y, Z) \mapsto C(Y, Z)$ defined by

$$p = L\xi + PN(\xi + \Phi(\xi)) \tag{2.4}$$

$$(T\Phi)(p) = L\Phi(\xi) + QN(\xi + \Phi(\xi)). \tag{2.5}$$

We employ the contraction mapping theorem to prove the existence of a fixed point of T. Note that, given any starting point $\Psi \in \Gamma$, the iterates of T generate a sequence of manifolds at times $t_n = nT$ which converge to a limit; thus our method is closely related to the Hadamard graph transform method as employed in [49].

We first show that the map T is well defined.

LEMMA 2.3. *For any $\Phi \in \Gamma$ and $p \in Y$ there exists a unique $\xi \in Y$ satisfying Eq.* (2.4).

Proof. Note that by $(G2)$ L^{-1} exists on Y. Thus we may consider the iteration

$$\xi^{k+1} = L^{-1}[p - PN(\xi^k + \Phi(\xi^k))]. \tag{2.6}$$

If $p \in Y$, then this map takes $\xi^k \in Y$ into $\xi^{k+1} \in Y$. For any two sequences $\{\xi^k\}$, $\{\eta^k\}$ generated by (2.6) we have, by $(G2)$, since $\Phi \in \Gamma$,

$$\begin{aligned} |\xi^{k+1} - \eta^{k+1}|_\gamma &\leqslant b^{-1}B\,|\xi^k + \Phi(\xi^k) - \eta^k - \Phi(\eta^k)|_\gamma \\ &\leqslant b^{-1}B(1+\delta)|\xi^k - \eta^k|_\gamma. \end{aligned}$$

By Condition $(C1)$ the mapping is a contraction and the existence of ξ given any $p \in Y$ follows. We also have the estimate

$$\begin{aligned} |\xi|_\gamma &\leqslant b^{-1}\,|p - PN(\xi + \Phi(\xi))|_\gamma \\ &\leqslant b^{-1}(|p|_\gamma + B). \quad \blacksquare \end{aligned} \tag{2.7}$$

Thus, by Lemma 2.3, $T\Phi$: $Y \to Z$ is well-defined. We now show that T maps $\Gamma(\delta, \varepsilon)$ into itself.

LEMMA 2.4. *The mapping T defined by* (2.5) *satisfies T: $\Gamma \mapsto \Gamma$. Furthermore if* (1.15) *holds, then if Φ satisfies* (2.1), *it follows that $T\Phi$ satisfies* (2.1).

Proof. From Eq. (2.5) and $(G1)$, $(G3)$ we have for all $p \in Y$ and $\Phi \in \Gamma$

$$\begin{aligned} |(T\Phi)(p)|_{\gamma} &\leqslant a\,|\Phi(\xi)|_{\gamma} + B \\ &\leqslant a\varepsilon + B. \end{aligned}$$

Thus we have, by $(C2)$, $\|T\Phi\|_{\Gamma} \leqslant \varepsilon$.

Let $p_1, p_2 \in Y$. From Lemma 2.3 there exists $\{\xi_i\}_{i=1}^{2}$ such that Eq. (2.5) is satisfied with $p = \{p_i\}_{i=1}^{2}$. Subtracting the two equations, we obtain

$$\begin{aligned} |(T\Phi)(p_1) - (T\Phi)(p_2)|_{\gamma} &\leqslant a\,|\Phi(\xi_1) - \Phi(\xi_2)|_{\gamma} + B\,|(\xi_1 - \xi_2) + \Phi(\xi_1) - \Phi(\xi_2)|_{\gamma} \\ &\leqslant [a\delta + B(1+\delta)]|\xi_1 - \xi_2|_{\gamma} \\ &= \theta\,|\xi_1 - \xi_2|_{\gamma}, \end{aligned}$$

where we have used $(G1)$, $(G3)$, $(C3)$ and the properties of $\Phi \in \Gamma$. From $(G2)$, $(G3)$ and (2.4) we have

$$\begin{aligned} b\,|\xi_1 - \xi_2|_{\gamma} &\leqslant |L(\xi_1 - \xi_2)|_{\gamma} \\ &\leqslant |p_1 - p_2|_{\gamma} + B(1+\delta)|\xi_1 - \xi_2|_{\gamma}. \end{aligned}$$

Using $(C1)$ we deduce that $\phi > 0$, and hence

$$|\xi_1 - \xi_2|_{\gamma} \leqslant \frac{1}{\phi}\,|p_1 - p_2|_{\gamma}.$$

Thus $(C3)$ implies

$$|(T\Phi)(p_1) - (T\Phi)(p_2)|_{\gamma} \leqslant \frac{\theta}{\phi}\,|p_1 - p_2|_{\gamma} \leqslant \delta\,|p_1 - p_2|_{\gamma}.$$

Hence $T\colon \Gamma \mapsto \Gamma$.

Now we prove that, under (1.15), if Φ satisfies (2.1), then so does $T\Phi$. Let $R \geqslant cr + B$ and $|p|_{\gamma} \geqslant R$. The iteration (2.4) shows that by $(G2)$ and $(G3)$

$$R \leqslant |p|_{\gamma} \leqslant |L\xi|_{\gamma} + B \leqslant c\,|\xi|_{\gamma} + B,$$

for $|p|_{\gamma} \geqslant R$. Thus $|\xi|_{\gamma} \geqslant (R - B)/c \geqslant r$, and hence $N(\xi + \Phi(\xi)) = 0$ by (1.15). Thus it follows that, in fact,

$$R \leqslant |p|_{\gamma} \leqslant |L\xi|_{\gamma} \leqslant c\,|\xi|_{\gamma},$$

and since $c < 1$, $|\xi|_\gamma \geqslant R$. Since Φ satisfies (2.1), then Eq. (2.5) gives

$$(T\Phi)(p) = L\Phi(\xi) = 0$$

for all $|p|_\gamma \geqslant R$. This completes the proof. ∎

Now we may show that the map T is a contraction on the space Γ.

LEMMA 2.5. *For any* $\Phi_1, \Phi_2 \in \Gamma$ *we have*

$$\|T\Phi_1 - T\Phi_2\|_\Gamma \leqslant \mu \|\Phi_1 - \Phi_2\|_\Gamma.$$

Proof. By Lemma 2.3, for any $p \in Y$ and $\{\Phi_i\}_{i=1}^2 \in \Gamma$ we can find $\{\xi_i\}_{i=1}^2$ such that for $i = 1, 2$

$$\begin{aligned} p &= PG(\xi_i + \Phi_i(\xi_i)) \\ (T\Phi_i)(p) &= QG(\xi_i + \Phi_i(\xi_i)). \end{aligned} \tag{2.8}$$

Using $(G1)$, $(G3)$ we have

$$|(T\Phi_1)(p) - (T\Phi_2)(p)|_\gamma \leqslant (a+B)|\Phi_1(\xi_1) - \Phi_2(\xi_2)|_\gamma + B\,|\xi_1 - \xi_2|_\gamma;$$

adding and subtracting $\Phi_2(\xi_1)$, using the triangle inequality and $(C3)$, we majorize the last inequality by

$$|(T\Phi_1)(p) - (T\Phi_2)(p)|_\gamma \leqslant (a+B)|\Phi_1(\xi_1) - \Phi_2(\xi_1)|_\gamma + \theta\,|\xi_1 - \xi_2|_\gamma. \tag{2.9}$$

Now, using $(G2)$ and Eq. (2.8), we have similarly that

$$\begin{aligned} b\,|\xi_1 - \xi_2|_\gamma \leqslant |L(\xi_1 - \xi_2)|_\gamma &\leqslant B\,|\xi_1 - \xi_2 + \Phi_1(\xi_1) - \Phi_2(\xi_2)|_\gamma \\ &\leqslant B(1+\delta)|\xi_1 - \xi_2|_\gamma + B\,|\Phi_1(\xi_1) - \Phi_2(\xi_1)|_\gamma. \end{aligned}$$

Thus since $\phi > 0$, by $(C1)$

$$|\xi_1 - \xi_2|_\gamma \leqslant \frac{B}{\phi}\,|\Phi_1(\xi_1) - \Phi_2(\xi_1)|_\gamma.$$

Returning to (2.9) and using $(C3)$, $(C4)$, we find

$$|(T\Phi_1)(p) - (T\Phi_2)(p)|_\gamma \leqslant \mu\,|\Phi_1(\xi_1) - \Phi_2(\xi_1)|_\gamma. \tag{2.10}$$

The result follows after taking the supremum over $\xi_1 \in Y$ and then $p \in Y$. ∎

Proof of Theorem 2.2. The existence of the manifold as the Graph(Φ) follows from Lemmas 2.3–2.5. That Φ satisfies (2.1) under (1.15) follows from Lemma 2.4.

To establish the exponential attraction of solutions to this manifold let $u_m = p_m + q_m$ be an arbitrary trajectory of (2.2), (2.3). Set

$$p = Lp_m + PN(p_m + \Phi(p_m)),$$
$$\Phi(p) = L\Phi(p_m) + QN(p_m + \Phi(p_m)).$$

Then using $(G1)$, $(G2)$ we have

$$\begin{aligned}|q_{m+1} - \Phi(p_{m+1})|_\gamma &\leqslant |q_{m+1} - \Phi(p)|_\gamma + |\Phi(p) - \Phi(p_{m+1})|_\gamma \\ &\leqslant (a+B)|q_m - \Phi(p_m)|_\gamma + \delta\, |p - p_{m+1}|_\gamma.\end{aligned}$$

However, by (2.2), we have

$$|p - p_{m+1}|_\gamma \leqslant B\, |q_m - \Phi(p_m)|_\gamma.$$

Thus by $(C4)$ we obtain

$$|q_{m+1} - \Phi(p_{m+1})|_\gamma \leqslant \mu\, |q_m - \Phi(p_m)|_\gamma$$

and

$$\begin{aligned}\operatorname{dist}_\gamma(u_{m+1}, \mathcal{M}) &\leqslant |(p_{m+1} + q_{m+1}) - (p_{m+1} + \Phi(p_{m+1}))|_\gamma \\ &\leqslant \mu^m\, |q_0 - \Phi(p_0)|_\gamma. \end{aligned} \tag{2.11}$$

Finally, we must show that Φ is independent 'of $\delta \in [\delta', K\delta']$, $\varepsilon \in [\varepsilon', K\varepsilon']$. Suppose $\Phi_{\varepsilon, \delta}$ (we temporally denote the dependence on ε, δ) is constructed for ε, δ taken in Definition 2.1 to be ε', δ', respectively. That is, $\Phi_{\varepsilon', \delta'}$ is a fixed point of T. Since the Conditions C hold for all ε, δ in these intervals, there is a $\Phi_{\varepsilon, \delta} \in \Gamma(\varepsilon, \delta)$ that is a fixed point for T. However, $\Phi_{\varepsilon', \delta'}$ is in the space $\Gamma(\varepsilon, \delta)$ and is a fixed point of T. Since the fixed points are unique, $\Phi_{\varepsilon', \delta'} = \Phi_{\varepsilon, \delta}$. ∎

3. An Approximation Theorem

In this section we prove the following result concerning the relationship between the invariant manifolds of (1.5) and (1.10)—the **Main Theorem I** of Section 1.

Theorem 3.1. *Suppose that the mappings* (1.5) *and* (1.10) *satisfy Assumptions G and G^h, respectively. If Conditions C' are satisfied with $K = (1 + C)\,C^3$, then there exist global functions $\Phi \in \Gamma$, $\Phi^h \in \Gamma^h$ such that $\mathcal{M} = Graph(\Phi)$ and $\mathcal{M}^h = Graph(\Phi^h)$ are attractive invariant manifolds for G, G^h satisfying* (1.7), (1.8) *and* (1.12), (1.13) *respectively. Furthermore if either* (1.15) *or* (1.16) *holds, then*:

(i) *for any* $p \in Y$ *there exists* $C(p) > 0$ *such that*

$$|(p + \Phi(p)) - (P^h p + \Phi^h(P^h p))|_\gamma \leqslant C(p)\, h;$$

(ii) *for any* $p^h \in Y^h$

$$|(Pp^h + \Phi(Pp^h)) - (p^h + \Phi^h(p^h))|_\gamma \leqslant C(p^h)\, h.$$

The theorem is proved through a sequence of lemmas. We prove first the case $c < 1$ where (1.15) holds (inertial manifolds); the case (1.16) where $b > 1$ (unstable manifolds) is a minor modification given at the end of the section. The basic idea of the proof is to use the uniform contraction principle. The technical difficulty which needs to be overcome is that Φ and Φ^h are graphs over different spaces; for this reason we introduce in the course of the proof the function Θ: $Y \mapsto Z$ defined by

$$\Theta(p) = Q\Phi^h(P^h p). \tag{3.1}$$

This function is a rotation of Φ^h to the space on which Φ acts.

We suppose throughout this section that Conditions G, G^h, C', hold without stating this explicitly. The existence of $\Phi \in \Gamma(\delta, \varepsilon)$ and $\Phi^h \in \Gamma^h(\delta, \varepsilon)$ follows from Theorem 2.2.

LEMMA 3.2. *The projections* P, Q, P^h, Q^h *defined by the eigenfunctions of* A *and* A^h *respectively, satisfy*

$$|PQ^h|_\gamma \leqslant Ch, \qquad |P^h Q|_\gamma \leqslant Ch$$

and

$$|QP^h|_\gamma \leqslant Ch, \qquad |Q^h P|_\gamma \leqslant Ch.$$

Proof. The result follows from applying (G^h4), (G^h6) to

$$\begin{aligned}
PQ^h &= P(E^h - P^h) = P(P - P^h)\, E^h,\\
P^h Q &= P^h(I - P) = P^h(P^h - P),\\
QP^h &= (I - P)\, P^h = (P^h - P)\, P^h,\\
Q^h P &= (E^h - P^h)\, P = E^h(P - P^h)\, P. \quad \blacksquare
\end{aligned}$$

LEMMA 3.3. *Let* (1.15) *hold. Then, for any* $p \in Y$ *there exists* $C(p) > 0$ *such that*

$$|(p + \Phi(p)) - (P^h p + \Phi^h(P^h p))|_\gamma \leqslant C(p)\, h.$$

Proof. We have, using (G^h4), Lemma 3.2 and (3.1)

$$\begin{aligned}
&|(p+\Phi(p))-(P^h p+\Phi^h(P^h p))|_\gamma \\
&\quad \leqslant |p-P^h p|_\gamma + |\Phi(p)-\Phi^h(P^h p)|_\gamma \\
&\quad \leqslant C\,|p|_\gamma\, h + |\Phi(p)-Q\Phi^h(P^h p)|_\gamma + |PQ^h\Phi^h(P^h p)|_\gamma \\
&\quad \leqslant C(|p|_\gamma + C\varepsilon)\, h + |\Phi(p)-\Theta(p)|_\gamma. \qquad (3.2)
\end{aligned}$$

We now show that if $\Phi^h\in\Gamma^h(\delta',\varepsilon')$, then $\Theta\in\Gamma$ for some $\varepsilon\geqslant\varepsilon'$, $\delta\geqslant\delta'$. Notice that, by (G^h6), (G^h7) and since $Q=I-P$,

$$\begin{aligned}
|\Theta(p)|_\gamma &\leqslant (1+C)|\Phi^h(P^h p)|_\gamma \\
&\leqslant C(1+C)|\Phi^h(P^h p)|_{h,\gamma} \leqslant C(1+C)\,\varepsilon';
\end{aligned}$$

similarly

$$\begin{aligned}
|\Theta(p_1)-\Theta(p_2)|_\gamma &\leqslant (1+C)|\Phi^h(P^h p_1)-\Phi^h(P^h p_2)|_\gamma \\
&\leqslant (1+C)\,C\,|\Phi^h(P^h p_1)-\Phi^h(P^h p_2)|_{h,\gamma} \\
&\leqslant (1+C)\,C\delta'\,|P^h p_1 - P^h p_2|_{h,\gamma} \\
&\leqslant (1+C)\,C^3\delta'\,|p_1-p_2|_\gamma.
\end{aligned}$$

Thus, $\Theta\in\Gamma$ with $\delta=(1+C)\,C^3\delta'$, $\varepsilon=(1+C)\,C\varepsilon'$. Moreover, from (G^h7), $C\geqslant 1$ and $(1+C)\,C\leqslant(1+C)\,C^3$.

We may now apply (2.10) from the proof of Lemma 2.5 to obtain

$$|T\Phi(p)-T\Theta(p)|_\gamma \leqslant \mu\,|\Phi(\xi)-\Theta(\xi)|_\gamma, \qquad (3.3)$$

where $T\Theta$ is defined by

$$\begin{aligned}
p &= PG(\xi+\Theta(\xi)) \\
T\Theta(p) &= QG(\xi+\Theta(\xi)).
\end{aligned} \qquad (3.4)$$

We now use (3.3) to estimate $|\Phi(p)-\Theta(p)|_\gamma$ in (3.2). Since Φ, Φ^h are fixed points for T and T^h respectively, we have, by (3.3)

$$\begin{aligned}
|\Phi(p)-\Theta(p)|_\gamma &= |T\Phi(p)-QT^h\Phi^h(P^h p)|_\gamma \\
&\leqslant \mu\,|\Phi(\xi)-\Theta(\xi)|_\gamma + |T\Theta(p)-QT^h\Phi^h(P^h p)|_\gamma. \qquad (3.5)
\end{aligned}$$

Thus we must estimate $|T\Theta(p)-QT^h\Phi^h(P^h p)|_\gamma$. To do this let

$$\begin{aligned}
\eta &= P^h G^h(P^h\xi+\Phi^h(P^h\xi)) \\
(T^h\Phi^h)(\eta) &= Q^h G^h(P^h\xi+\Phi^h(P^h\xi)).
\end{aligned} \qquad (3.6)$$

We have, using the properties of Φ^h, the fact that $Q = I - P$ and that $T^h \Phi^h = \Phi^h$,

$$\begin{aligned}
&|T\Theta(p) - QT^h\Phi^h(P^h p)|_\gamma \\
&\quad \leqslant |T\Theta(p) - QT^h\Phi^h(\eta)|_\gamma + (1+C)|T^h\Phi^h(\eta) - T^h\Phi^h(P^h p)|_\gamma \\
&\quad \leqslant |T\Theta(p) - QT^h\Phi^h(\eta)|_\gamma + (1+C)\, C^2\delta\, |\eta - P^h p|_\gamma.
\end{aligned} \tag{3.7}$$

Since $QT\bullet = T\bullet$, $Qp = 0$, we majorize the first term in (3.7), adding and subtracting $Q\eta$, by

$$|T\Theta(p) - QT^h\Phi^h(\eta)|_\gamma \leqslant |Q(p + T\Theta(p)) - Q(\eta + T^h\Phi^h(\eta))|_\gamma + |QP^h\eta|_\gamma. \tag{3.8}$$

Similarly the second term is proportional to a term majorized by

$$|\eta - P^h p|_\gamma \leqslant |P^h(\eta + T^h\Phi^h(\eta)) - P^h(p + T\Theta(p))|_\gamma + |P^h QT\Theta(p)|_\gamma, \tag{3.9}$$

where we have used $P^h\eta = \eta$, and $P^h T^h = 0$. We have from (3.4), (3.6) and (G^h5)

$$\begin{aligned}
&|(\eta + T^h\Phi^h(\eta)) - (p + T\Theta(p))|_\gamma \\
&\quad \leqslant C(\xi)[h + |(\xi - P^h\xi) + (Q\Phi^h(P^h\xi) - \Phi^h(P^h\xi))|_\gamma].
\end{aligned}$$

Furthermore, we have by Lemma 3.2,

$$|Q\Phi^h(P^h\xi) - \Phi^h(P^h\xi)|_\gamma = |P\Phi^h(P^h\xi)|_\gamma = |PQ^h\Phi^h(P^h\xi)|_\gamma \leqslant C^2\varepsilon h$$

and from (G^h4)

$$|\xi - P^h\xi|_\gamma = |P\xi - P^h\xi|_\gamma \leqslant Ch\, |\xi|_\gamma,$$

so that

$$|(\eta + T^h\Phi^h(\eta)) - (p + T\Theta(p))|_\gamma \leqslant C(\xi)\, h. \tag{3.10}$$

The last terms in (3.8), (3.9) can be bounded by using Lemma 3.2. Thus we obtain from (3.7), using (3.8), (3.9), (3.10)

$$|T\Theta(p) - QT^h\Phi^h(P^h p)|_\gamma \leqslant C(\xi)\, h.$$

Returning to (3.5), we have that

$$|\Phi(p) - \Theta(p)|_\gamma \leqslant \mu\, |\Phi(\xi) - \Theta(\xi)|_\gamma + C(\xi)\, h, \tag{3.11}$$

where p and ξ are related through (3.4). Recall from Theorem 2.2 that there is a unique ξ for every $p \in Y$. Since (1.15) holds and since Θ is constructed as in (3.1), we have from Lemma 2.4 that $\Phi(p)$ and $\Theta(p)$ are identically zero for $|p|_\gamma \geqslant 2R$, for h sufficiently small. We take the supremum over all ξ which yields p: $|p|_\gamma \leqslant 2R$ to obtain, from (3.11),

$$\|\Phi - \Theta\|_{\Gamma} = \sup_{|p| \leqslant 2R} |\Phi(p) - \Theta(p)|_\gamma \leqslant \mu \|\Phi - \Theta\|_{\Gamma} + C(R)\, h.$$

Thus it follows that

$$\|\Phi - \Theta\|_{\Gamma} \leqslant (1-\mu)^{-1}\, C(R)\, h. \tag{3.12}$$

This is the estimate needed in (3.2) to prove the Lemma. ∎

LEMMA 3.4. *Let* (1.15) *hold. Then, for any* $p^h \in Y^h$ *there exists* $C(p^h) > 0$ *such that*

$$|(p^h + \Phi^h(p^h)) - (Pp^h + \Phi(Pp^h))|_\gamma \leqslant C(p^h)\, h.$$

Proof. Set

$$u^h = p^h + \Phi^h(p^h), \qquad u = Pp^h + \Phi(Pp^h)$$

and

$$w^h = P^h P p^h + \Phi^h(P^h P p^h).$$

Applying Lemma 3.3 we obtain $|w^h - u|_\gamma \leqslant C(Pp^h)\, h$. Using the properties of Φ^h and (G^h4), (G^h6) we find

$$\begin{aligned} |u^h - w^h|_\gamma &\leqslant (1+\delta)|p^h - P^h P p^h|_\gamma = (1+\delta)|P^h(P^h - P)\, p^h|_\gamma \\ &\leqslant (1+\delta)\, C^2 h\, |p^h|_\gamma. \end{aligned}$$

Hence $|u^h - u|_\gamma \leqslant C(p^h)\, h$ as required. ∎

Proof of Theorem 3.1, *Case* (1.15). The proof follows from Lemmas 3.3 and 3.4. ∎

The following corollary concerns only the behavior of the functions Φ, Φ^h as opposed to points on the inertial manifold.

COROLLARY 3.5. *For the intertial manifold case* (1.15), *and for h sufficiently small,* $\exists K>0$ *such that*

$$\sup_{p\in Y} |\Phi(p)-\Phi^h(P^h p)|_\gamma \leqslant Kh$$

$$\sup_{p^h\in Y^h} |\Phi(Pp^h)-\Phi^h(p^h)|_\gamma \leqslant Kh.$$

Proof. The first inequality follows from (3.12):

$$\sup_{p\in Y} |\Phi(p)-\Phi^h(P^h p)|_\gamma$$
$$\leqslant \|\Phi-\Theta^h\|_{\Gamma}+\sup_{p\in Y} |P\Phi^h(P^h p)|_\gamma \leqslant (1-\mu)^{-1}\, C(R)\, h + Ch\varepsilon.$$

The second follows from the conclusion of Lemma 3.4:

$$|\Phi(Pp^h)-\Phi^h(p^h)|_\gamma \leqslant C(p^h)\, h + |(P-P^h)\, p^h|_\gamma$$
$$\leqslant C(p^h)\, h + Ch\, |p^h|_\gamma.$$

Also from assumption (G^h4) we have for h sufficiently small, $|Pp^h|_\gamma \geqslant |p^h|_\gamma/2$. Thus from (2.1), for $|p^h|_\gamma \geqslant 4R$, $\Phi(Pp^h)=0$, $\Phi^h(p^h)=0$. The result follows. ∎

We conclude this section with a modification of Theorem 3.1 in the case (1.16) appropriate to the construction of unstable manifolds.

Proof of Theorem 3.1, *Case* (1.16). The proof is identical to the proof in the case (1.15) except that (2.1) no longer holds. Hence (3.11) is still true but the reasoning thereafter is different.

By (3.4) we have that

$$b\, |\xi|_\gamma \leqslant |L\xi|_\gamma \leqslant |p|_\gamma + B, \tag{3.13}$$

where $b>1$. We now consider $|p|_\gamma$ in two different regimes; let $\chi \geqslant B/(b-1)$.

(i) if $B/(b-1) \leqslant |p|_\gamma \leqslant \chi$, then (3.13) gives

$$b\, |\xi|_\gamma \leqslant \chi + (b-1)\,\chi = b\chi \Rightarrow |\xi|_\gamma \leqslant \chi.$$

Thus, by (3.11)

$$\sup_{B/b-1 \leqslant |p|_\gamma \leqslant \chi} |\Phi(p)-\Theta(p)|_\gamma \leqslant \mu \sup_{|\xi|_\gamma \leqslant \chi} \{|\Phi(\xi)-\Theta(\xi)|_\gamma + C(\xi)\, h\}. \tag{3.14}$$

(ii) if $|p|_\gamma \leqslant B/(b-1)$, then by (3.13) we have $|\xi|_\gamma \leqslant B/(b-1)$. Thus

$$\sup_{|p|_\gamma \leqslant B/(b-1)} |\Phi(p)-\Theta(p)|_\gamma \leqslant \mu \sup_{|\xi|_\gamma \leqslant B/(b-1)} \{|\Phi(\xi)-\Theta(\xi)|_\gamma + C(\xi)\, h\}. \tag{3.15}$$

Since $\chi \geqslant B/(b-1)$, combining (3.14) and (3.15) gives

$$\sup_{|p|_\gamma \leqslant \chi} |\Phi(p)-\Theta(p)|_\gamma \leqslant \mu \sup_{|\xi|_\gamma \leqslant \chi} \{|\Phi(\xi)-\Theta(\xi)|_\gamma + C(\xi)\, h\}.$$

and hence, it follows that

$$\sup_{|p|_\gamma \leqslant \chi} |\Phi(p)-\Theta(p)|_\gamma \leqslant (1-\mu)^{-1} \sup_{|\xi|_\gamma \leqslant \chi} C(\xi)\, h \leqslant C(\chi)\, h.$$

Equation (3.2) then gives the desired result to obtain (i) of Theorem 3.1; part (ii) follows by an argument identical to that in Lemma 3.4 for the case where (1.15) holds. ∎

4. Existence of Inertial Manifolds for PDEs

In this section we show that the assumptions and conditions of Section 2 hold for mappings derived from a class of nonlinear dissipative PDEs which includes the Kuramoto-Sivashinsky equation, the Cahn-Hilliard equation, the Ginzburg-Landau equation and certain reaction-diffusion equations. In so doing we will construct inertial manifolds for these equations. To do this it will be necessary to show that the invariant manifold constructed in Section 2 for the map (1.5) also yields an invariant manifold for the partial differential equations whose semigroup define the map (1.5).

All the equations mentioned above can be written as an evolution on a Hilbert space, denoted by X, as

$$\frac{du}{dt} + Au = R(u). \tag{4.1}$$

We define the eigenvalues and eigenvectors of A, the spaces X^γ and projections P, Q, and subspaces Y, Z as in Section 1. Throughout this section and section 5, we make the following **Assumptions** E about this Equation:

($E1$) the operator A is an unbounded self-adjoint linear operator with compact inverse and eigenvalues $\{\lambda_i\}$ satisfying $0<\lambda_1 \leqslant \lambda_2 \leqslant \cdots$; thus we may take $\tilde{A} = A$ in (1.2);

$(E2)$ $\exists \gamma \geqslant 0$, $\beta \in [0, 1)$ and $E(\sigma) > 0$ such that the nonlinear function R: $X^{\gamma} \to X^{\gamma - \beta}$ satisfies

$$|R(u)|_{\gamma - \beta} \leqslant E(\sigma) \qquad \forall u \in \mathscr{B}_{\gamma}(0, \sigma)$$
$$|R(u) - R(v)|_{\gamma - \beta} \leqslant E(\sigma)|u - v|_{\gamma} \qquad \forall u, v \in \mathscr{B}_{\gamma}(0, \sigma);$$

$(E3)$ the equation generates a Lipschitz continuous semigroup $S_0(t)$: $X^{\gamma} \mapsto X^{\gamma}$. The resulting dynamical system is dissipative in the sense that there exists $\mathscr{B}_{\gamma}(0, \rho)$, so that for every $r > 0$ there exists a $T = T(r, \rho) \geqslant 0$ such that $S_0(t)\, \mathscr{B}_{\gamma}(0, r) \subset \mathscr{B}_{\gamma}(0, \rho)$ for all $t \geqslant T$.

Thus, due to the absorbing property, we may truncate the nonlinear term for large $|u|_{\gamma}$ as in [25]. This allows the construction of an attractive invariant manifold globally representable as a graph. However, the resulting modified equation is identical to the original equation within the absorbing set and so the intersection of the graph with the absorbing set defines an attractive invariant manifold for the original equation.

Specifically, we introduce $\theta \in C^{\infty}(\mathbb{R}^+, [0, 1])$, a fixed smooth function satisfying:

$$\begin{aligned} \theta(s) &= 1, & 0 \leqslant s \leqslant \rho^2, \\ \theta(s) &= 0, & s \geqslant 2\rho^2, \\ |\theta'(s)| &\leqslant 2 & \forall s \geqslant 0. \end{aligned} \tag{4.2}$$

Now define

$$F(u) = \theta(|u|_{\gamma}^2)\, R(u),$$

and consider the equation

$$\frac{du}{dt} + Au = F(u). \tag{4.3}$$

This equation has the same behavior as (4.1) for all solutions inside the ball $\mathscr{B}_{\gamma}(0, \rho)$ and thus, to understand the long-time dynamics of (4.1) it is sufficient to study (4.3).

Using the properties of $R(u)$ and the definition of θ it follows that $\exists K_1, K_2 \geqslant 0$ such that the nonlinear operator $F(u)$ satisfies the estimates

$$\begin{aligned} |F(u)|_{\gamma - \beta} &\leqslant K_1 & \forall u \in X^{\gamma}; \\ |F(u) - F(v)|_{\gamma - \beta} &\leqslant K_2\, |u - v|_{\gamma} & \forall u, v \in X^{\gamma}. \end{aligned} \tag{4.4}$$

Using this it is straightforward to prove global existence of a solution to (4.3) and to define a Lipschitz continuous semigroup $S(t)$: $X^{\gamma} \mapsto X^{\gamma}$. Thus we may now proceed to construct an inertial manifold for (4.3) by using the theory from Section 2.

LEMMA 4.1. *Suppose that for any* K_3, $K_4 > 0$ *there exists an integer* $q_0 > 0$ *such that*

$$\lambda_{q+1}^{1-\beta} \geqslant K_3, \qquad \lambda_{q+1} - \lambda_q \geqslant K_4 \lambda_{q+1}^{\beta}$$

for all $q \geqslant q_0$. *Then for any* K, ε', $\delta' > 0$, *there exists* $T > 0$, *integer* $q_0 > 0$ *and* $\mu \in (0, 1)$ *such that, if P is the projection associated with the first q eigenfunctions of A and* $q \geqslant q_0$, *and*

$$L = e^{-AT}, \qquad N(u) = \int_0^T L(T-s)\, F(S(s)\, u)\, ds,$$

then Assumptions G, Conditions C′ and (1.15) *hold for the map* $G(u) = Lu + N(u)$.

Proof. We define $\lambda = \lambda_q$, $\Lambda = \lambda_{q+1}$. Then L satisfies $(G1)$, $(G2)$ with

$$b = e^{-\lambda T}, \qquad a = e^{-\Lambda T}, \qquad c = e^{\lambda_1 T}.$$

We set $\Lambda T = \sigma$ and consider the time T map of the flow. In general, it is not possible to use a time T map of the flow with $T \to 0$. This is because $L = I + O(T)$ and $N(u) = O(T^{1-\beta})$. Thus, to leading order in T, $(L - I)$ does not dominate the nonlinear term as $T \to 0$. We prove *a posteriori* that the manifold we construct is invariant for all t. From the definition of $N(u)$ we have, using Theorem 2.6.13 of [52], a constant $M > 0$:

$$\begin{aligned} |N(u)|_\gamma &\leqslant \int_0^T |A^\beta L(T-s)\, A^{\gamma-\beta} F(S(s)\, u)|\, ds \\ &\leqslant \int_0^T \frac{MK_1}{(T-s)^\beta}\, ds \leqslant \frac{T^{1-\beta}}{1-\beta} MK_1. \end{aligned} \tag{4.5}$$

Similarly, using $(E2)$, the continuity of the semigroup $S(t)$, and the construction of F,

$$|N(u) - N(v)|_\gamma \leqslant \frac{T^{1-\beta}}{1-\beta} CMK_2\, |u - v|_\gamma,$$

for some $C > 0$. Thus, $N(u)$ satisfies $(G3)$ with

$$B = T^{1-\beta} K_5,$$

where $K_5 = M \max\{K_1, CK_2\}/(1-\beta)$.

We verify Conditions C. Let $\varepsilon, \delta > 0$, $\sigma \in (0, \infty)$, $\mu \in (e^{-\sigma}, 1)$ be given. Define K_3, K_4 and T by

$$T = \frac{\sigma}{\Lambda}$$

$$K_3 = \max\left\{\frac{e^{\sigma} K_5 \sigma^{1-\beta}(1+\delta)}{\mu - e^{-\sigma}}, \frac{(1+\sigma) K_5}{\varepsilon \sigma^{\beta}}\right\},$$

$$K_4 = \frac{K_5(1+\delta)^2 e^{\sigma}}{\delta \sigma^{\beta}}.$$

If we require $\Lambda^{1-\beta} \geqslant K_3 \geqslant e^{\sigma} K_5 \sigma^{1-\beta}(1+\delta)/\mu$, then

$$e^{(\lambda - \Lambda) T} e^{\Lambda T} K_5 T^{1-\beta}(1+\delta) \leqslant \mu,$$

or $b^{-1} B(1+\delta) \leqslant \mu$ as required to establish $(C1)$.

Moreover, if we require $\Lambda^{1-\beta} \geqslant K_3 \geqslant (1+\sigma) K_5/(\sigma^{\beta}\varepsilon)$, then

$$(1 + \Lambda T) K_5 T \frac{\Lambda^{\beta}}{\sigma^{\beta}} \leqslant \varepsilon \Lambda T.$$

Adding ε to both sides and dividing by $1 + \Lambda T$, we obtain

$$\frac{\varepsilon}{1 + \Lambda T} + K_5 T^{1-\beta} \leqslant \varepsilon.$$

Since $e^{-x} \leqslant 1/(1+x)$ for all $x > 0$, we have $a\varepsilon + B \leqslant \varepsilon$ and (C2) is established.

To prove $(C3)$ we require

$$\Lambda - \lambda \geqslant K_4 \Lambda^{\beta} \geqslant K_5(1+\delta)^2 e^{\sigma} \frac{\Lambda^{\beta}}{\delta \sigma^{\beta}}.$$

Since $e^x - 1 \geqslant x$ for positive x, we may bound $(\Lambda - \lambda) T$ by $e^{(\Lambda - \lambda) T} - 1$. Multiplying the previous inequality by T and using $\sigma = \Lambda T$, we find

$$\delta e^{-\Lambda T} + K_5(1+\delta)^2 T^{1-\beta} \leqslant \delta e^{-\lambda T}.$$

Since $(1+\delta)^2 = (1+\delta) + \delta(1+\delta)$, this last inequality becomes $\delta a + (1+\delta) B \leqslant \delta b - \delta(1+\delta) B$ which is $(C3)$.

$(C4)$ is obtained by requiring

$$\Lambda^{1-\beta} \geqslant K_3 \geqslant K_5 \sigma^{1-\beta} \frac{(1+\delta)}{\mu - e^{-\sigma}}.$$

Rearranging this gives $K_5 T^{1-\beta}(1+\delta)+e^{-AT}\leqslant\mu$. Using the definitions of a, B, this last expression becomes $a+(1+\delta)B\leqslant\mu$ which is $(C4)$.

It remains to establish (1.15). Clearly $c<1$ since $\lambda_1>0$. Let r be a positive number satisfying $br-B\geqslant\sqrt{2}\,\rho$. Recall the definitions (1.4) of $L(t)$ and $N(u,t)$. From the identity

$$p(t)=L(t)\,p(0)+\int_0^t PL(t-s)\,F(u(s))\,ds,$$

where again P is the projection onto $\operatorname{span}\{\varphi_1, ..., \varphi_q\}$, and the above bounds on L and N for $0\leqslant t\leqslant T$, we obtain

$$|p(t)|_\gamma\geqslant b\,|p(0)|_\gamma-B$$

for $t\in[0,T]$. Thus if $|p(0)|_\gamma\geqslant r$, we have $|p(t)|_\gamma\geqslant\sqrt{2}\,\rho$ for $0\leqslant t\leqslant T$. By construction of P and Q as orthogonal projections, we have $|u(t)|_\gamma^2\geqslant|p(t)|_\gamma^2\geqslant 2\rho^2$; hence, by the definition of θ, and hence F, we have

$$N(u):=\int_0^T L(T-s)\,F(S(s)\,u)\,ds=0$$

provided $|Pu|_\gamma\geqslant r$. Thus (1.15) holds.

Now to establish Conditions C' note that q_0 depends on K_3 and K_4, and hence ε, δ. Taking the supremum of q_0 over all $\varepsilon\in[\varepsilon', K\varepsilon']$, $\delta\in[\delta', K\delta']$ yields q_0 such that conditions C' hold. ∎

By Theorem 2.2, Lemma 4.1 shows that the time T flow of the semigroup for (4.3) has an attractive invariant manifold. We show now that this manifold is in fact an inertial manifold for the continuous Equation (4.1).

THEOREM 4.2. *Under Assumptions E and the assumptions of Lemma* 4.1 *on the spectrum of A, Eq.* (4.1) *has an inertial manifold: an exponentially attracting, positively invariant, finite dimensional manifold which can be represented as a graph* $\Phi\colon Y\mapsto Z$ *within* $\mathscr{B}_\gamma(0,\rho)$.

Proof. It remains to show that the manifold $\mathscr{M}$ constructed as a consequence of Lemma 4.1 is in fact invariant for the underlying partial differential equation, rather than just the time T flow of the semigroup. To do this set $\Omega=S(\tau)\,\mathscr{M}$ for some $\tau\in(0,T)$. As before, $\mathscr{M}=Graph(\Phi)$. We will show that, for all τ sufficiently small, $\Omega\equiv\mathscr{M}$ and hence that $\mathscr{M}$ is invariant for the Equation (4.3). By the construction of (4.3) from (4.1), the required result then follows.

Notice that $S(T)\,\Omega=S(T)\,S(\tau)\,\mathscr{M}=S(\tau)\,S(T)\,\mathscr{M}=S(\tau)\,\mathscr{M}=\Omega$ so that Ω is invariant under the time T discrete map. In addition, we can show that Ω is the graph of a global function. Recall the definitions of $L(t)$ and

$N(u, t)$ given in (1.4); by applying the method of proof of Lemma 2.3 it follows that, for every $p \in Y$ there exists a unique ξ so that

$$p = L(\tau)\,\xi + PN(\xi + \Phi(\xi), \tau),$$

for any $\tau \in (0, T)$. Thus Ω can be expressed as a graph $q = \Psi(p)$ where $\Psi\colon Y \mapsto Z$ is given by

$$\Psi(p) := L(\tau)\,\Phi(\xi) + QN(\xi + \Phi(\xi), \tau)$$

Assume that $\Phi \in \Gamma(\delta', \varepsilon')$. As in (4.5) we have

$$\begin{aligned} |\Psi|_\gamma &\leqslant e^{-A\tau}\varepsilon' + \frac{\tau^{1-\beta}}{1-\beta} MK_1 \\ &\leqslant e^{A(T-\tau)} a\varepsilon' + B\left(\frac{\tau}{T}\right)^{1-\beta}. \end{aligned}$$

Notice that for $\tau = 0$, $\tau = T$, $|\Psi|_\gamma \leqslant \varepsilon'$. In general, for $0 \leqslant \tau \leqslant T$ there is some $\eta \geqslant 0$ such that

$$|\Psi|_\gamma \leqslant \varepsilon' + \eta.$$

In a similar manner, one obtains

$$|\Psi(p_1) - \Psi(p_2)|_\gamma \leqslant (\delta' + \eta)|p_1 - p_2|_\gamma$$

for $0 \leqslant \tau \leqslant T$. Note that, by continuity, η can be made arbitrarily small by requiring τ to be small. Thus we see that, for any $\eta > 0$ there exists a $\tau^*(\eta) > 0$ such that

$$\Psi \in \Gamma(\varepsilon' + \eta, \delta' + \eta)$$

for all $\tau \in (0, \tau^*(\eta))$. Thus we chose M so large that Conditions C, verified in the previous lemma, hold for all $\delta \in [\delta', \delta' + \eta]$, $\varepsilon \in [\varepsilon', \varepsilon' + \eta]$.

For such δ, ε, $\Psi \in \Gamma(\delta, \varepsilon)$ and we may again apply Lemma 2.3 to obtain that for every $p \in Y$ there exists a $p_0 \in Y$ such that

$$\begin{aligned} p &= Lp_0 + PN(p_0 + \Psi(p_0)), \\ q &= L\Psi(p_0) + QN(p_0 + \Psi(p_0)). \end{aligned}$$

However, since $S(T)\,\Omega = \Omega$, $u = p + q \in \Omega$ and $\Omega = \mathit{Graph}(\Psi)$, we must have $q = \Psi(p)$. Thus Ψ is a fixed point of the map T constructed in (2.5), and by the uniqueness of the fixed point under Conditions C', it follows that $\Psi \equiv \Phi$. One may now repeat the argument on the intervals $t \in (k\tau, (k+1)\tau)$ for integer $k \geqslant 1$. It follows that $\mathcal{M} = Graph(\Phi)$ is an invariant manifold for Eq. (4.3) for all time.

To see that this manifold exponentially attracts all solutions let $t > 0$ and u_0 be given. Set $t_n(s) = nT + s$, $s \in [0, T]$. Then from (2.11)

$$dist(u(t_n), \mathcal{M}) = dist(S(nT)\, u(s), \mathcal{M}) \leqslant \mu^n C(|u(s)|_\gamma).$$

Since $u(s)$ depends continuously on $u(0)$ for all $s \in (0, T)$, the result follows. ∎

5. The Spectral Method and Inertial Manifolds

Recall Assumptions $E2$ which we make throughout this section. In this section we apply a spectral approximation to Eq. (4.1) based on the eigenfunctions of A. Similar results have been obtained in [25], [26] where the analysis was based on the Lyapunov-Perron method, but here we also consider the effect of time discretization (a time discretization alone is studied in [12]).

Let $P^N: X \mapsto X$ be defined by the orthogonal projection onto the $span\{\varphi_1, ..., \varphi_N\}$, where $\{\varphi_j\}$ are the eigenfunctions of A; $A\varphi_j = \lambda_j \varphi_j$, where we suppose that A is self-adjoint. We also set $Q^N = I - P^N$ and $W = P^N X$. We recall that $Y = PX$, $Z = QX$. Then the spectral approximation of (4.1) is to find a solution v in the space W solving

$$\begin{aligned} \frac{dv}{dt} + Av &= P^N R(v), \\ v(0) &= P^N u_0, \end{aligned} \tag{5.1}$$

where $R(v)$ satisfies the Assumption $E2$ of Section 4. We also suppose throughout this section that $0 \leqslant \beta < 1$.

Unless further assumptions are made about $R(v)$ it is not possible in general to show that (5.1) is dissipative. However, for the PDEs mentioned at that the beginning of Section 4, this is the case. Indeed, one technique used to show that these PDEs are dissipative is to first show that their Galerkin approximation, (5.1), has this property and then pass to the limit as $N \to \infty$ (for a collection of such results see [58] which also contains references to the original works). We therefore assume that (5.1) is dissipative. We mention here that in general to show that approximating schemes, such as the ones under consideration in this paper, are dissipative requires some care. This question is considered in, for example [39, 21, 22, 18, 35, 36].

As with the PDE (4.1) we truncate the nonlinear term outside the ball $B_\gamma(0, \rho)$. That is, we consider on the space W the equation

$$\begin{aligned} \frac{dv}{dt} + Av &= P^N F(v), \\ v(0) &= P^N u_0, \end{aligned} \tag{5.2}$$

where $F(v) = \theta(|v|_\gamma)\, R(v)$. If (5.1) is dissipative then (5.2) has the same long-time dynamics as (5.1). Note that (5.2) is the spectral approximation of equation (4.3), which has an inertial manifold $\mathcal{M}$, globally representable as a graph $\Phi(\bullet)$: $Y \mapsto Z$.

THEOREM 5.1. *Equation (5.2) generates a Lipschitz semigroup $S^N(t)$ on X^γ. In addition, there exists $\kappa > 0$, $K > 0$ and $N_0 > 0$ such that (5.2) has an inertial manifold $\mathcal{M}^N$, representable as the graph of a function Φ^N: $Y \mapsto Q^N Z$, for all $N \geqslant N_0$. Moreover, for any $\iota > 0$ there exists $k(\iota) > 0$ such that*

$$dist\{S^N(t)\, v(0), \mathcal{M}^N\} \leqslant k(\iota)\, e^{-\kappa t} \qquad \forall t \geqslant 0,$$

for any $v(0) \in B_\gamma(0, \iota)$, and the functions Φ, Φ^N satisfy for all $N \geqslant N_0$

$$\sup_{p \in Y} |\Phi(p) - \Phi^N(p)|_\gamma \leqslant \frac{KA^{1-\beta}}{\lambda_N^{1-\beta}}.$$

We prove this below, but first we state a corollary. We wish to compare the manifolds $\mathcal{M}^N$ and $\mathcal{M}$ inside the absorbing set (inside the cut-off region where $\theta = 1$ in (4.2)). Since the construction of both $\mathcal{M}^N$ and $\mathcal{M}$ involves a prepared equation, it is necessary to localize the comparison of $\mathcal{M}^N$ and $\mathcal{M}$ in order to compare them for the original equations. To this end recall definitions (1.17) and (1.18).

COROLLARY 5.2. *Suppose (4.1) has an absorbing ball $B_\gamma(0, \rho)$ for all $\rho \in (\rho_c, \infty)$ and that the cut-off function θ in (4.2) is chosen with $\rho = \rho_0 > \rho_c$. Then for all $r < \rho_0$ there exist $r' \geqslant \rho_c$ and $N_0 \geqslant q_0$, the dimension of the inertial manifold for (4.1), such that*

$$dist(\mathcal{M}_{loc}(0, r), \mathcal{M}^N_{loc}(0, r')) \leqslant \frac{KA^{1-\beta}}{\lambda_{N+1}^{1-\beta}},$$

$$dist(\mathcal{M}^N_{loc}(0, r), \mathcal{M}_{loc}(0, r')) \leqslant \frac{KA^{1-\beta}}{\lambda_{N+1}^{1-\beta}}$$

for all $N \geqslant N_0$.

We emphasize that, provided (5.1) is dissipative, the local manifolds $\mathcal{M}_{loc}$, $\mathcal{M}^N_{loc}$ are positively invariant attracting manifolds for the unprepared equations (4.1) and (5.1) respectively.

Proof of Theorem 5.1. The existence of a Lipschitz semigroup and an inertial manifold follows exactly as in Theorem 4.2. To establish the convergence we set

$$L^h = \exp(AT), \qquad N^h(v) = \int_0^T L^h(T-s)\, P^N F(S^N(s)\, v)\, ds,$$

where S^N is the semigroup for (5.2). We must verify the Assumptions G^h for the map $G^h(u) = L^h u + N^h(u)$. $(G^h 1)-(G^h 3)$ are immediate and follow as in the proof of Theorem 4.2. The space V^h is given by $V^h = P^N X = E^N X = W$ and $P = P^h$. Thus $(G^h 4)$ is trivially satisfied. $(G^h 6)$, $(G^h 7)$ follow with $C = 1$. It remains to verify $(G^h 5)$.

Let $\alpha(t) = P^N u(t) - v(t)$, $\beta(t) = Q^N u(t)$, where $v(t)$ is the solution of (5.2) with $v(0) = P^N u(0)$. The total error is $e(t) = \alpha(t) + \beta(t)$. Since the solution of Eq. (4.3) may be written as

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} F(u(s))\, ds \tag{5.3}$$

with a similar expression for solutions of (5.2), we have

$$\alpha(t) = e^{-tA} \alpha(0) + \int_0^t e^{-(t-s)A} [P^N(F(u(s)) - F(v(s)))]\, ds. \tag{5.4}$$

Thus

$$|\alpha(t)|_\gamma \leqslant |\alpha(0)|_\gamma + \int_0^t |A^\beta e^{-(t-s)A} A^{\gamma-\beta} P^N(F(u(s)) - F(v(s)))|\, ds. \tag{5.5}$$

From (4.4) and $(E2)$ we find

$$|\alpha(t)|_\gamma \leqslant |\alpha(0)|_\gamma + K \int_0^t \frac{|e(s)|_\gamma}{(t-s)^\beta},$$

where we have used the estimate $|A^\eta e^{-tA}| \leqslant C t^{-\eta}$ which holds for $\eta \geqslant 0$, $t > 0$. Therefore,

$$\begin{aligned} |e(t)|_\gamma &\leqslant |\alpha(t)|_\gamma + |\beta(t)|_\gamma \\ &\leqslant |Q^N u(t)|_\gamma + |\alpha(0)|_\gamma + K \int_0^t \frac{|e(s)|_\gamma}{(t-s)^\beta}. \end{aligned}$$

To estimate the first term we set $u(t)=\sum_{j=1}^{\infty} u_j\varphi_j$ and we use Corollary 7.3 in the appendix. We conclude that

$$
\begin{aligned}
|Q^N u(t)|_\gamma^2 &= \sum_{j=N+1}^{\infty} \lambda_j^{2\gamma} u_j^2 \leqslant \frac{1}{\lambda_{N+1}^{2(1-\beta)}} \sum_{j=N+1}^{\infty} \lambda_j^{2(1+\gamma-\beta)} u_j^2 \\
&\leqslant \frac{1}{\lambda_{N+1}^{2(1-\beta)}} |u(t)|_{\gamma-\beta+1}^2 \\
&\leqslant \frac{C}{\lambda_{N+1}^{2(1-\beta)} t^{2(1-\beta)}}.
\end{aligned} \tag{5.6}
$$

Using the Henry-Gronwall lemma ([31]), we obtain

$$
|u(T)-v(T)|_\gamma \leqslant K\,|P^N(v(0)-u(0))|_\gamma + \frac{K\Lambda^{1-\beta}}{\lambda_{N+1}^{1-\beta}},
$$

for $0<t\leqslant T$ noting that T was chosen such that $T=\sigma/\Lambda$ (see Section 4). (G^h5) follows.

Conditions C' hold by Lemma 4.1. Thus Theorem 3.1 applies. Moreover, since $P=P^h$ in Assumptions G^h, the proof of Theorem 3.1 is simplified greatly and (G^h5) gives the rate of convergence of the inertial manifolds. That is, $h=(\Lambda/\lambda_N)^{1-\beta}$ in (G^h5). Since the graphs $\mathcal{M}=Graph(\Phi)$ *and* $\mathcal{M}^N=Graph(\Phi^h)$, which are pointwise close are constructed for the prepared equations, and since the prepared and true equations agree within the absorbing sets, the localized convergence result, Corollary 5.2, follows. Moreover, an argument similar to that used to show that $\mathcal{M}$ is invariant for $S(t)$ for all $t>0$ in Theorem 4.2, shows that $\mathcal{M}^N$ is invariant for $S^N(t)$. ∎

As an application of the convergence of the inertial manifolds recall that the PDE (4.3) restricted to its inertial manifold gives the finite-dimensional inertial form

$$
\frac{dp}{dt}+Ap=PF(p+\Phi(p)) \tag{5.7}
$$

which has the same long-time dynamics as the original equation (1.1). Similarly, the Galerkin scheme restricted to its inertial manifold gives

$$
\frac{dp^N}{dt}+Ap^N=PF(p^N+\Phi^N(p^N))
$$

where $p^N(t)=Pv(t)$. This system has the same long-time dynamics as the original Galerkin system (5.1) and has the same dimension M as (5.7) as

$N \to \infty$. Thus Theorem 5.1 shows (as in [25], [26] for this case) that the two inertial forms are C close. If one show C^1 closeness of the inertial manifolds, then since both equations are ODEs, one may apply the results of [53] (and the references therein) to conclude that certain hyperbolic structures of the attractor are preserved by the this Galerkin scheme for N sufficiently large. These possibilities are studied in [41].

Now we consider a semi-implicit time discretization of (5.2). Rather than proving or assuming that this time discretization of (5.1) is dissipative, we consider the direct approximation of the prepared equation, (4.3), (5.2) (the dissipativity of time discretazations is considered in [21] [35]). Consider

$$\begin{aligned} v_{n+1} - v_n + \Delta t A v_{n+1} &= \Delta t P^N F(v_n), \\ v_0 &= P^N u_0. \end{aligned} \tag{5.8}$$

We will also need to assume that $F(u) = \theta(|u|_\gamma)\, R(u)$ satisfies

$$\begin{aligned} |F'(u)\, v|_{\gamma-\beta} &\leqslant K\, |v|_\gamma, \\ |F'(u_1)\, v - F'(u_2)\, v|_{\gamma-\beta} &\leqslant K\, |u_1 - u_2|_\gamma\, |v|_\gamma \end{aligned} \tag{5.9}$$

for all $u, v, u_1, u_2 \in X^\gamma$ with $0 \leqslant \beta < 1$. These follow from the assumption that

$$|R'(u)\, v|_{\gamma-\beta} \leqslant E(\sigma)|v|_\gamma, \qquad |(R'(u_1) - R'(u_2))\, v|_{\gamma-\beta} \leqslant E(\sigma)|u_1 - u_2|_\gamma\, |v|_\gamma$$

for all $u, v, u_1, u_2 \in \mathscr{B}_\gamma(0, \sigma)$ and is satisfied by the PDEs mentioned in the previous section.

THEOREM 5.3. *Equation* (5.8) *generates a Lipschitz semigroup* $S^{N, \Delta t}(n)$ *on* X^γ. *In addition, there exists* $\kappa > 0$, $N_0 > 0$, $\tau_0 > 0$ *such that* (5.8) *has an inertial manifold* $\mathscr{M}^{N, \Delta t}$, *representable as the graph of a function* $\Phi^{N, \Delta t}: PX \mapsto Q^N X$ *for all* $\Delta t \leqslant \tau_0$, $N \geqslant N_0$. *Moreover, for any* $\iota > 0$ *there exists* $k(\iota) > 0$ *such that*

$$dist\{S^{N, \Delta t}(n) v(0), \mathscr{M}^{N, \Delta t}\} \leqslant k(\rho)\, e^{-\kappa n \Delta t} \qquad \forall n \geqslant 0,$$

for any $v(0) \in B_\gamma(0, \iota)$ *and for any* $\varepsilon > 0$ *there exists* $K_\varepsilon > 0$ *such that the functions* Φ, $\Phi^{N, \Delta t}$ *satisfy*

$$\sup_{p \in Y} |\Phi(p) - \Phi^{N, \Delta t}(p)|_\gamma \leqslant K_\varepsilon \Lambda \left(\frac{\lambda_N}{\Lambda}\right)^\varepsilon \Delta t + K \left(\frac{\Lambda}{\lambda_{N+1}}\right)^{1-\beta},$$

$$\forall \Delta t \leqslant \tau_0,\ N \geqslant N_0.$$

Proof. From (5.8) it follows that for any $\Delta t > 0$

$$v_{n+1} = (I + \Delta t A)^{-1} [v_n + \Delta t P^N F(v_n)]. \tag{5.10}$$

An iteration of this expression gives

$$v_n = (I + \Delta t A)^{-n} v_0 + \Delta t \sum_{j=1}^{n} (I + \Delta t A)^{j-n-1} P^N F(v_{j-1}).$$

If $\{u_n\}_0^\infty$ also satisfies (5.8), then $w_n := u_n - v_n$ satisfies

$$w_n = (I + \Delta t A)^{-n} w_0 + \Delta t \sum_{j=1}^{n} (I + \Delta t A)^{j-n-1} P^N (F(u_{j-1}) - F(v_{j-1})).$$

Using (4.4) and Lemma 7.6, we obtain

$$|w_{n+1}|_\gamma \leqslant |w_0|_\gamma + \Delta t K \sum_{j=1}^{n} \frac{1}{t_{n-j+1}^\beta} |w_{j-1}|_\gamma,$$

where $t_{n-j+1} = (n-j+1)\,\Delta t$. Applying the discrete Gronwall lemma, [17], we conclude

$$|w_{n+1}|_\gamma \leqslant K(T) |w_0|_\gamma.$$

Thus the semigroup defined by $S^{N,\Delta t}(n)$ defined by (5.10) is Lipschitz.

Now we define

$$L^h = (I + \Delta t A)^{-m}, \qquad N^h(u) = \Delta t \sum_{j=1}^{m} (I + \Delta t A)^{j-m-1} P^N F(S^{N,\Delta t}(j)\, u) \tag{5.11}$$

with $m\Delta t = T$ and $G^h = L^h + N^h$. We must verify the Assumptions G^h for the map G^h. Using the same argument directly above, the assumptions (4.4) on $F(u)$ and Lemma 7.6 in the appendix, we have that

$$\begin{aligned} |N^h(u)|_\gamma\, \Delta t \sum_{j=1}^{m} \frac{K}{((m-j+1)\,\Delta t)^\beta} \\ \leqslant K \int_0^T \frac{dt}{(T-t)^\beta} \leqslant K \frac{T^{1-\beta}}{1-\beta} \end{aligned}$$

A similar calculation shows that

$$|N^h(u) - N^h(v)|_\gamma \leqslant K \frac{T^{1-\beta}}{1-\beta} |u - v|_\gamma.$$

As above all of the assumptions on G^h are immediate except for (G^h5). We claim that for all $u_0, v_0 \in X$

$$|u(n\,\Delta t) - v_n|_\gamma \leqslant K\,|P^N(u_0 - v_0)|_\gamma + K_\varepsilon \Lambda \left(\frac{\lambda_N}{\Lambda}\right)^\varepsilon \Delta t + K\left(\frac{\Lambda}{\lambda_{N+1}}\right)^{1-\beta}, \quad (5.12)$$

where $K_\varepsilon \to \infty$ as $\varepsilon \to 0$. For the proof of this claim we essentially follow the proof given in [47] for the finite element case. We remark that since we are measuring the error of the prepared equations, our estimates are not as sharp as for direct approximation of Eq. (4.1). This is due to the fact that the only Lipschitz property the prepared nonlinear term satisfies is $|F(u) - F(v)|_{\gamma-\beta} \leqslant K\,|u - v|_\gamma$.

We set

$$\begin{aligned} e_n = v_n - u_n &= (v_n - P^N u_n) + (P^N u_n - u_n) \\ &:= \theta_n - q_n, \end{aligned}$$

where $q_n = Q^N u(n\,\Delta t)$ and is estimated in (5.6). Now we estimate θ_n. Using (5.8) we see that θ_j solves

$$\frac{\theta_j - \theta_{j-1}}{\Delta t} + A\theta_j = -P^N(F(v_{j-1}) - F(u_j)) - \omega_j,$$

where

$$\omega_j = P^N\left(\frac{u_j - u_{j-1}}{\Delta t} - \frac{du(j\,\Delta t)}{dt}\right)$$

and $u_j = P^N u(j\,\Delta t)$. Summing, we have

$$\theta_m = (I + \Delta t A)^{-m}\,\theta_0 - \sum_{j=1}^{m} \Delta t(I + \Delta t A)^{j-m-1}\,[P^N(F(v_{j-1}) - F(u_j)) + \omega_j], \quad (5.13)$$

where again $m\,\Delta t = T$. Set

$$R_1 := \left|\sum_{j=1}^{m} \Delta t A^\gamma (I + \Delta t A)^{j-m-1}\,\omega_j\right|$$

and $n_1 = [m/2] + 1$. Then

$$\begin{aligned} R_1 \leqslant& \sum_{j=1}^{n_1} \Delta t\,|A(I + \Delta t A)^{j-m-1}|\,|A^\varepsilon A^{\gamma-1-\varepsilon}\omega_j| \\ &+ \sum_{j=n_1+1}^{m} \Delta t\,|A^{1-\varepsilon}(I + \Delta t A)^{j-m-1}|\,|A^\varepsilon A^{\gamma-1}\omega_{j+1}|. \end{aligned}$$

Using Lemma 7.5 in the appendix, we have for $\eta > -\gamma$

$$|\omega_j|_{-\eta} = \left| \frac{1}{\Delta t} \int_{t_{j-1}}^{t_j} (t - t_{j-1}) P^N u_{tt}(t)\, dt \right|_{-\eta}$$
$$\leqslant \frac{K}{\Delta t\, t_{j-1}^{2-\gamma-\eta}} \int_{t_{j-1}}^{t_j} (t - t_{j-1})\, dt \leqslant \frac{K\, \Delta t}{2 t_j^{2-\gamma-\eta}}.$$

In the first term we take $\eta = 1 + \varepsilon - \gamma$ and in the second we take $\eta = 1 - \gamma$. Now we use Lemma 7.6 in the appendix and the bound $|A^\varepsilon P^N| \leqslant \lambda_N^\varepsilon |P^N|$ to bound R_1 by

$$R_1 \leqslant K \sum_{j=1}^{n_1} \frac{\Delta t^2 \lambda_N^\varepsilon}{t_{m-j+1}} \frac{1}{t_j^{1-\varepsilon}} + K \sum_{j=n_1+1}^{m} \frac{1}{t_{m-j+1}^{1-\varepsilon}} \frac{\Delta t^2 \lambda_N^\varepsilon}{t_j}$$
$$\leqslant K_\varepsilon\, \Delta t \lambda_N^\varepsilon T^{\varepsilon-1}.$$

For the other term in Eq. (5.13) we have

$$R_2 := \left| \sum_{j=1}^{m} \Delta t A^\gamma (I + \Delta t A)^{j-m-1} P^N (F(v_{j-1}) - F(u_j)) \right|$$
$$\leqslant \sum_{j=1}^{m} K\, \Delta t\, |A^\beta (I + \Delta t A)^{j-m-1}|\, |v_{j-1} - u_j|_\gamma.$$

However,

$$|v_{j-1} - u_j|_\gamma \leqslant |v_{j-1} - P^N u_{j-1}|_\gamma + |P^N(u_{j-1} - u_j) - q_j|_\gamma$$
$$\leqslant |\theta_{j-1}|_\gamma + \Delta t \lambda_N^\varepsilon \left| P^N \frac{(u_j - u_{j-1})}{\Delta t} \right|_{\gamma-\varepsilon} + |q_j|_\gamma.$$

Furthermore, by the mean value theorem and Eq. (7.4) in the appendix we have

$$\left| P^N \frac{(u_j - u_{j-1})}{\Delta t} \right|_{\gamma-\varepsilon} \leqslant \max_{(j-1)\Delta t \leqslant t \leqslant j\Delta t} \left| \frac{du}{dt} \right|_{\gamma-\varepsilon} \leqslant \frac{K}{t_{j-1}^{1-\varepsilon}} \leqslant \frac{K}{t_j^{1-\varepsilon}}. \tag{5.14}$$

Using (5.6) we conclude

$$R_2 \leqslant K \sum_{j=1}^{m} \frac{\Delta t}{t_{m-j+1}^\beta} \left[|\theta_{j-1}|_\gamma + \frac{\Delta t \lambda_N^\varepsilon}{t_j^{1-\varepsilon}} + \frac{1}{\lambda_{N+1}^{1-\beta} t_j^{1-\beta}} \right]$$
$$\leqslant K \sum_{j=1}^{m} \frac{\Delta t}{t_{m-j+1}^\beta} |\theta_{j-1}|_\gamma + \frac{K_\varepsilon \lambda_N^\varepsilon \Delta t}{T^{\beta-\varepsilon}} + \frac{K}{\lambda_{N+1}^{1-\beta}}.$$

Combining this with the previous estimate, we have, from (5.13),

$$|\theta_m|_\gamma \leqslant \left(|\theta_0|_\gamma + 2K_\varepsilon \Delta t \frac{\lambda_N^\varepsilon}{T^{1-\varepsilon}} + \frac{K}{\lambda_{N+1}^{1-\beta}}\right) + K\Delta t \sum_{j=1}^{m} \frac{|\theta_j|_\gamma}{t_{m-j+1}^\beta},$$

where we have used $t_{m-j+1}^{-\beta} \leqslant 2^\beta t_{m-(j-1)+1}^{-\beta}$. Noting that $\beta < 1$ and that we may assume $T < 1$ without loss of generality by appropriate choice of q, we conclude from the discrete Gronwall lemma, [17] and Eq. (5.6), that

$$|e_m|_\gamma = |v_m - u_m|_\gamma \leqslant K\,|v_0 - P^N u_0|_\gamma + 2K_\varepsilon \Delta t \frac{\lambda_N^\varepsilon}{T^{1-\varepsilon}} + \frac{K}{\lambda_{N+1}^{1-\beta}} + \frac{K}{T^{1-\beta}\lambda_{N+1}^{1-\beta}}.$$

Recalling that $TA = \sigma$, Eq. (5.12) follows. Conditions C' hold by Lemma 4.1 and Theorem 3.1 gives the convergence of the inertial manifolds. Again since $P = P^h$ in Assumptions G^h, (G^h5) controls the rate of convergence of the inertial manifolds. We remark that $\mathcal{M}^{N,\Delta t}$ is an invariant manifold for the discrete semigroup $S^{N,\Delta t}(m)$. An argument similar to that used to show that $\mathcal{M}$ is invariant for $S(t)$ for all $t > 0$ in Theorem 4.2, shows that $\mathcal{M}^{N,\Delta t}$ is invariant for $S^{N,\Delta t}(1)$. ▮

6. The Finite Element Method and Inertial Manifolds

In this section we consider the equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} &= f(u) \qquad (x, t) \in (0, 1) \times (0, \infty) \\ u(0, t) = u(1, t) &= 0 \qquad t > 0. \end{aligned} \tag{6.1}$$

We assume that

$$f \in C^2(\mathbb{R}, \mathbb{R}), \qquad \limsup_{|u| \to \infty} \frac{f(u)}{u} \leqslant 0, \qquad \exists K > 0\colon f'(u) \leqslant K. \tag{6.2}$$

We define $A\colon \mathcal{D}(A) \mapsto X$ by

$$A = -\alpha \frac{\partial^2}{\partial x^2},$$

where $\mathcal{D}(A) = H^2((0, 1)) \cap H_0^1((0, 1))$, $X = L^2((0, 1))$. We also set $V := X^{1/2}$ and define $R\colon V \mapsto V$ by

$$(R(u))(x) := f(u(x)).$$

Note that $X^{1/2}=H_0^1((0,1))$ and that $|\bullet|_{1/2}$ is equivalent to the standard $H_0^1((0,1))$ norm. Thus, Eq. (6.1) takes the form of Eq. (4.1). The spaces X^α, Y, Z and the norms $|\cdot|_\alpha$ are as defined in Section 1. In the following the definition of a gradient semigroup is taken from [28]. We can now verify Conditions G and C' for Eq. (6.1).

LEMMA 6.1. *Equation* (6.1) *satisfies Assumptions E and generates a* C^1 *gradient semigroup* $S_0(t)$ *on* V. *Furthermore there exists* $\kappa>0$ *and* $\rho>0$ *such that* (6.1) *has an inertial manifold* $\mathcal{M}$, *locally representable as the graph of a function* Φ: $Y\mapsto Z$ *within* $\mathcal{B}_{1/2}(0,\rho)$ *for* q *sufficiently large* (*to satisfy Lemma* 4.1). *Moreover, for any* $\iota>0$ *there exists* $k(\iota)>0$, *such that*

$$\mathrm{dist}(S_0(t)\,u_0,\mathcal{M})\leqslant k(\iota)\,e^{-\kappa t}\qquad \forall t\geqslant 0, \tag{6.3}$$

for any $u_0\in\mathcal{B}_{1/2}(0,\iota)$.

Proof. We apply Theorem 4.2. (E1) follows simply from the properties of $-\Delta$ on an interval. Straightforward calculation shows that R: $V\to V$ is Lipschitz so that (E2) may be established with $\gamma=12$ and $\beta=0$. Notice also that the equation has the global Lyapunov functional

$$V(\phi)=\int_0^1\left[\frac{\alpha}{2}\,\phi^2-g(\phi)\right]dx \tag{6.4}$$

with $g'(\phi)=f(\phi)$; specially all solutions of (6.1) satisfy

$$\frac{d}{dt}\,V(u(t))=-|u_t(t)|^2. \tag{6.5}$$

The existence and continuity of the semigroup follows from [28] (p. 76) and [52] (Theorem 6.3.1) with $\gamma=1/2$, $\beta=0$. Theorem 4.3.1 of [28] gives the existence of a global attractor. Hence we deduce the existence of $\rho_c>0$ so that $\mathcal{B}_{1/2}(0,\rho)$ is an absorbing set in V for any $\rho\in(\rho_c,\infty)$; this follows since any neighborhood in V containing the attractor is an absorbing set. Thus (E3) holds.

For later reference we define, for θ given by (4.2),

$$\begin{aligned} &F(u)=\theta(|u|_{1/2}^2)\,R(u),\\ &L=e^{-AT},\qquad N(u)=\int_0^T L(T-s)\,F(S(s)\,u)\,ds \end{aligned} \tag{6.6}$$

where $S(s)$ is the semigroup for the prepared Eq. (4.3). Applying Theorem 4.2 gives the existence and attraction result for $\mathcal{M}$ since the eigenvalues of A are $\lambda_n=\alpha\pi^2n^2$ so that the conditions of Lemma 4.1 hold with $\beta=0$. ∎

We now approximate (6.1) by the finite element method. Let $\{V^h\}_{h>0}$ denote a family of finite dimensional subspaces of V comprised of piecewise linear functions with respect to the domain (0, 1) with maximum partition size h. We assume that the partition of (0, 1) is such that

$$|E^h|_{1/2} \leqslant C, \tag{6.7}$$

where $E^h \colon X \mapsto V^h$ is defined by

$$(E^h u, \chi) = (u, \chi) \qquad \forall \chi \in V^h, u \in X;$$

That this can be achieved is proved in [11]. The approximation $u^h(t) \in V^h$ for the solution $u(t) \in V$ of (6.1) solves

$$\begin{aligned} (u_t^h, \chi) + \alpha(u_x^h, \chi_x) &= (f(u^h), \chi) \qquad \forall \chi \in V^h \\ u^h(0) &= u_0^h \in V^h. \end{aligned} \tag{6.8}$$

Moreover, we define A^h by

$$(A^h \psi, \chi) = \alpha(\psi_x, \chi_x) \qquad \forall \psi, \chi \in V^h,$$

and $R^h \colon V^h \mapsto V^h$ by

$$(R^h(\psi))(x) = E^h f(\psi(x)).$$

Since A^h is self-adjoint, we may define the spaces $X^{h,\alpha}$, Y^h, Z^h and the norms $|\cdot|_{h,\alpha}$ as described in Section 1. It is well-known (see [37]) that there exists a $C > 1$ such that

$$C^{-1} |v|_{1/2} \leqslant |v|_{h,1/2} \leqslant C |v|_{1/2} \qquad \forall v \in V^h. \tag{6.9}$$

In summary, Eq. (6.8) takes the form

$$\frac{du^h}{dt} + A^h u^h = R^h(u^h). \tag{6.10}$$

We now show that the approximation (6.10) of (6.1) satisfies Assumption G^h and Conditions C', and hence show the existence of the manifold $\mathcal{M}^h$ close to $\mathcal{M}$.

LEMMA 6.2. *Equation* (6.10) *satisfies Assumptions E and generates a C^1 gradient semigroup $S_0^h(t)$ on V^h. Furthermore there exists $\rho > 0$, $\kappa > 0$, $h_0 > 0$ and an inertial manifold $\mathcal{M}^h$, locally representable as the graph of a function*

$\Phi^h: Y^h \mapsto Z^h$ within $\mathscr{B}_{1/2}(0, \rho)$, for h sufficiently small. Moreover, for any $\iota > 0$ there exists $k(\iota) > 0$ such that

$$dist(S_0^h(t)\, u_0, \mathcal{M}^h) \leqslant k(\iota)\, e^{-\kappa t} \qquad \forall t \geqslant 0,$$

for all $u_0 \in \mathscr{B}_{h,(1/2)}(0, \iota)$.

Proof. We aim to put (6.10) in the framework of Section 4 and apply Theorem 4.2. Since A is self-adjoint, A^h is also so that $(E1)$ follows. Furthermore, from (6.9) and (6.7)

$$\begin{aligned} |R^h(u) - R^h(v)|_{h,1/2} &\leqslant C^2\, |f(u) - f(v)|_{1/2} \\ &\leqslant C^3 E(\rho) |u - v|_{h,1/2} \qquad \forall u, v \in \mathscr{B}_{h,(1/2)}(0, \rho). \end{aligned}$$

Hence $R^h: V^h \mapsto V^h$ is locally Lipschitz, $(E2)$ holds and the existence of a local solution follows. Setting $\chi = u_t^h$ in Eq. (6.8) shows that all solutions of (6.8) satisfy

$$\frac{d}{dt} V(u^h(t)) = -|u_t^h(t)|^2. \tag{6.11}$$

where V is the Lyapunov functional (6.4); the existence of a solution defined for all time follows as in [28] (p. 76). Following the method of proof in Lemma 6.1 an absorbing set $\mathscr{B}_{1/2}(0, \rho)$ may be constructed and $(E3)$ is obtained; note that, without loss of generality, we may assume that $\rho \in (\rho_c, \infty)$ as for the absorbing sets for $S_0(t)$. Note, furthermore, that we have taken the absorbing set in $X^{1/2}$ rather than $X^{h, 1/2}$ for convenience; norm equivalence (6.9) of these spaces for elements in V^h allows us to do this.

Again, for later reference, we define

$$\begin{aligned} &F^h(u) = \theta(|u|_{1/2}^2)\, R^h(u), \\ &L^h = e^{-A^h T}, \qquad N^h(u) = \int_0^T L^h(T-s)\, F^h(S^h(s)\, u)\, ds \end{aligned} \tag{6.12}$$

where θ is given by (4.2) and $S^h(s)$ is the semigroup for the prepared equation

$$\frac{du^h}{dt} + A^h u^h = F^h(u^h)$$

constructed from (6.10), using the function θ to define F^h.

To show the existence of $\mathcal{M}^h$ we apply Theorem 4.2 and it is required simply to establish the requisite spectral properties. Using the fact that A^h

is self-adjoint and that the eigenvalues of A^h converge to those of A [37], we deduce that the spectral conditions can be satisfied for any $K_3, K_4 > 0$. The existence of $\mathcal{M}^h$ follows. ▌

We are now in a position to apply Theorem 3.1. The next theorem describes the behavior of the functions Φ, Φ^h.

THEOREM 6.3. *Let Φ, Φ^h be as given in Lemmas 6.1 and 6.2. There exists a $h_c > 0$ such that for all $h \leqslant h_c$:*

(i) *for any $p \in Y$ there exists $C(p) > 0$ such that*

$$|(p + \Phi(p)) - (P^h p + \Phi^h(P^h p))|_\gamma \leqslant C(p)\, h;$$

(ii) *for any $p^h \in Y^h$*

$$|(Pp^h + \Phi(Pp^h)) - (p^h + \Phi^h(p^h))|_\gamma \leqslant C(p^h)\, h.$$

Furthermore, $\exists K > 0$:

$$\sup_{p \in Y} |\Phi(p) - \Phi^h(P^h p)|_\gamma \leqslant Kh,$$

$$\sup_{p^h \in Y^h} |\Phi(Pp^h) - \Phi^h(p^h)|_\gamma \leqslant Kh.$$

As in the spectral case above we wish to compare the invariant manifolds $\mathcal{M}^h$ and $\mathcal{M}$ in side the absorbing balls. It will again be necessary to localize the comparison of $\mathcal{M}^h$ and $\mathcal{M}$ in order to compare them for the original equations. To this end recall definitions (1.17) and (1.18).

COROLLARY 6.4. *Suppose Eq. (6.1) and (6.8) have absorbing balls $\mathcal{B}_{1/2}(0, \rho)$, for $\rho \in (\rho_c, \infty)$, and that (6.1) and (6.8) are prepared with radius $\rho = \rho_0 > \rho_c$. Then for all $r < \rho_0$ there exist $r' \geqslant \rho_c$, $h_0 > 0$ and $c > 0$ such that*

$$dist(\mathcal{M}_{loc,\,1/2}(0, r), \mathcal{M}^h_{loc,\,1/2}(0, r')) \leqslant ch \qquad \forall h \leqslant h_0,$$

$$dist(\mathcal{M}^h_{loc,\,1/2}(0, r), \mathcal{M}_{loc}(0, r')) \leqslant ch \qquad \forall h \leqslant h_0.$$

Proof of Theorem 6.3. We apply Theorem 3.1 to $G(u) = Lu + N(u)$ and $G^h(u) = L^h(u) + N^h(u)$ defined by (6.6) and (6.12) respectively. We denote the eigenvalues of A by $\{\lambda_j^0\}_{j=0}^{\infty}$ and those of A^h by $\{\lambda_j^h\}_{j=0}^{\infty}$. We set

$$\lambda^0 = \lambda_q^0, \qquad \lambda^h = \lambda_q^h, \qquad \Lambda^0 = \lambda_{q+1}^0, \qquad \Lambda^h = \lambda_{q+1}^h.$$

We have that $\lambda_i^h \to \lambda_i^0$ as $h \to 0$. Thus we define

$$\Lambda = \inf_{0 \leqslant h \leqslant h_0} \Lambda^h, \qquad \lambda = \sup_{0 \leqslant h \leqslant h_0} \lambda^h, \qquad \lambda_1 = \inf_{0 \leqslant h \leqslant h_0} \lambda_1^h.$$

Further, we let $\Lambda T = \alpha$ and define

$$b = e^{-\lambda T}, \qquad a = e^{-\Lambda T}, \qquad c = e^{-\lambda_1 T}.$$

By construction $(G1)$, $(G^h 1)$, $(G2)$ and $(G^h 2)$ are satisfied for $h \leqslant h_0$. The bounds $(G3)$, $(G^h 3)$ follow as in the proof of Lemmas 6.1 and 6.2. Furthermore, following Lemma 4.1 we deduce that Conditions C' hold provided that for any K_3, $K_4 > 0$ there exists an integer $q > 0$ such that $\Lambda \geqslant K_3$ and $\Lambda - \lambda \geqslant K_4$. Since Λ, λ can be made arbitrarily close to Λ^0, λ^0 by the choice of h_0, this is clearly true.

It remains to establish $(G^h 4)$–$(G^h 7)$. $(G^h 4)$ can be proved by a modification of Lemma 3.5 in [48]. Condition $(G^h 7)$ is a standard properties of the finite element method [37]–see (6.9); $(G^h 5)$ and the bounds on $|P|_\gamma$, $|P^h|_\gamma$ in $(G^h 6)$ are proved in [48]. Though in [47] $(G^h 5)$ is proven for the unprepared equation, the only Lipschitz property of the nonlinear term f used is $|f(u) - f(v)| \leqslant K\,|u - v|_{1/2}$ which is satisfied by the prepared equation. Indeed, once the equations are prepared this is the only Lipschitz property satisfied by $F(u) = \theta R(u)$. The bound on $|E^h|_\gamma$ in $(G^h 6)$ follows from the assumptions on the mesh partition described above–see (6.7).

Thus Theorem 3.1 applies. Since the graphs $\mathcal{M} = Graph(\Phi)$ *and* $\mathcal{M}^h = Graph(\Phi^h)$ which are pointwise close are constructed for the prepared equations and since the prepared and true equations agree within the absorbing sets, the localized convergence result, Corollary 6.4 follows. ∎

We now consider the effects of temporal discretization. Specifically, we apply the backward Euler approximation to (6.10). We obtain

$$(u^h_{n+1} - u^h_n) + \Delta t A^h u^h_{n+1} = \Delta t R^h(u^h_{n+1}). \tag{6.13}$$

For simplicity we will consider (6.131) under the limit process h, $\Delta t \to 0$ with

$$\frac{\Delta t}{h} = d. \tag{6.14}$$

For the discrete semigroup generated by (6.13) we will define an absorbing set to be a ball $\mathscr{B}_{h,\gamma}(0, r)$ which all solutions of (6.13) starting within a bounded set $\mathscr{B}_{h,1/2}(0, \sigma) \subset V^h$ enter and remain inside after a finite number of steps $N = N(\sigma, r)$. Using the methods in [16], [18] it may be shown that for $K\Delta t \leqslant 1$, where K is given by (6.2),

$$\frac{V(u^h_{n+1}) - V(u^h_n)}{\Delta t} \leqslant -\frac{1}{2}\,\frac{|u^h_{n+1} - u^h_n|^2}{\Delta t^2}. \tag{6.15}$$

Here $V: X^{1/2} \mapsto \mathbb{R}$ is given by (6.4). This is the discrete analog of Eq. (6.5) and enables us to prove the existence of an absorbing set for the scheme (6.13).

THEOREM 6.5. *Equation* (6.13) *generates a Lipschitz continuous semigroup*, $S_0^{h,\Delta t}(n)$, *on* V^h *with absorbing set* $\mathscr{B}_{1/2}(0, \rho)$ *for any* $\rho \in (\rho_c, \infty)$. *Furthermore there exists* $\kappa > 0$, $\tau_0 > 0$, $h_c > 0$ *and an inertial manifold* $\mathscr{M}^{h,\Delta t}$, *locally representable as the graph of the function* $\Phi^{h,\Delta t}$ *within* $\mathscr{B}_{1/2}(0, \rho)$, *for all* $\Delta t \leqslant \tau_0$, $h \leqslant h_c$. *Moreover, for any* $\iota > 0$ *there exists* $k(\iota) > 0$ *such that*

$$dist\{S^{h,\Delta t}(n)\, u_0^h, \mathscr{M}^{h,\Delta t}\} \leqslant k(\iota)\, e^{-\kappa n \Delta t} \qquad \forall n \geqslant 0,$$

for all $u_0 \in B_{1/2}(0, \iota)$. *Finally*,

(i) *for any* $p \in Y$ *there exists* $C(p) > 0$ *such that*

$$|(p + \Phi(p)) - (P^h p + \Phi^{h,\Delta t}(P^h p))|_\gamma \leqslant C(p)\, h;$$

(ii) *for any* $p^h \in Y^h$

$$|(Pp^h + \Phi(Pp^h)) - (p^h + \Phi^{h,\Delta t}(p^h))|_\gamma \leqslant C(p^h)\, h.$$

Furthermore, $\exists K > 0$:

$$\sup_{p \in Y} |\Phi(p) - \Phi^{h,\Delta t}(P^h p)|_\gamma \leqslant Kh$$

$$\sup_{p^h \in Y^h} |\Phi(Pp^h) - \Phi^{h,\Delta t}(p^h)|_\gamma \leqslant Kh.$$

Suppose Eq. (6.1) *and* (6.13) *have absorbing balls* $\mathscr{B}_{1/2}(0, \rho)$, *for* $\rho \in (\rho_c, \infty)$, *and inertial manifolds* $\mathscr{M}$ *and* $\mathscr{M}^{h,\Delta t}$ *respectively. In addition, suppose that* (6.1) *and* (6.13) *are prepared with radius* $\rho = \rho_0 > \rho_c$. *Then for all* $r < \rho_0$ *there exist* $r' \geqslant \rho_c$ *and* $h_0 > 0$ *such that*

$$dist(\mathscr{M}_{loc,(1/2)}(0, r), \mathscr{M}^{h,\Delta t}_{loc,(1/2)}(0, r')) \leqslant ch \qquad \forall h \leqslant h_0,$$

$$dist(\mathscr{M}^{h,\Delta t}_{loc,(1/2)}(0, r), \mathscr{M}_{loc}(0, r')) \leqslant ch \qquad \forall h \leqslant h_0.$$

Proof. Note that, under (6.2), the methods of [35] may be used to show that, for any $u_n^h \in V^h$, $\Delta t > 0$ there is a $u_{n+1}^h \in V^h$ satisfying (6.13). Moreover, for any two sequences $\{u_n^h\}$, $\{v_n^h\}$ satisfying (6.13), we have

$$(u_{n+1}^h - v_{n+1}^h, \chi) - (u_n^h - v_n^h, \chi) + \Delta t \alpha((u_{n+1}^h - v_{n+1}^h)_x, \chi_x)$$
$$= \Delta t(R^h(u_{n+1}^h) - R^h(v_{n+1}^h), \chi)$$

for all $\chi \in V^h$. If we suppose that at some n, $u_n^h = v_n^h$, then setting $\chi = u_{n+1}^h - v_{n+1}^h$ and noting that from (6.2) we have

$$(R^h(u_{n+1}^h) - R^h(v_{n+1}^h), \chi) \leqslant K\, |u_{n+1}^h - v_{n+1}^h|^2,$$

it follows from the previous equation that for $K\Delta t \leqslant 1/2$ the solution sequence $\{u_n^h\}$ is unique. In addition, a similar calculation yields the Lipschitz continuity result

$$|u_{n+1}^h - v_{n+1}^h|^2 \leqslant (1 - 2\Delta t K)^{-1}\, |u_n^h - v_n^h|^2.$$

Thus the existence of a Lipschitz semigroup $S^{h,\Delta t}(n)\colon V^h \mapsto V^h$ is established, where $u_n^h = S^{h,\Delta t}(n)\, u_0^h$. We may now apply Proposition 2.5 of [18] to deduce from (6.15) the existence of a compact connected global attractor $\mathscr{A}^{h,\Delta t} \subset V^h$ for $S_0^{h,\Delta t}(n)$. Using the same argument as in Lemma 6.1 we construct the absorbing set $\mathscr{B}_{1/2}(\rho)$, for any $\delta > 0$, where, without loss of generality, $\rho \in (\rho_c, \infty)$.

As before we now consider the prepared equation

$$(u_{n+1}^h - u_n^h) + \Delta t A^h u_{n+1}^h = \Delta t F^h(u_{n+1}^h), \tag{6.16}$$

where $F^h(u)$ is given by (6.12). Now define

$$L^h = (I + \Delta t A^h)^{-m}, \qquad N^h(u) = \Delta t \sum_{j=1}^{m} (I + \Delta t A^h)^{j-m-1}\, F^h(S^{h,\Delta t}(j)\, u), \tag{6.17}$$

where $S^{h,\Delta t}(j)$ is the semigroup for the prepared equation.

Using Lemma 7.6 in the Appendix we have from (6.7), (6.9), the boundedness of F in V and the choice $\gamma = 1/2$, $\beta = 0$

$$\begin{aligned} |N^h(u)|_{h,1/2} &\leqslant \Delta t \sum_{j=1}^{m} |(I + \Delta t A^h)^{j-m-1}\, F^h(s^{h,\Delta t}(j)\, u)|_{h,(1/2)} \\ &\leqslant C\Delta t \sum_{j=1}^{m} |F^h(s^{h,\Delta t}(j)\, u)|_{h,(1/2)} \\ &\leqslant Cm\, \Delta t. \end{aligned}$$

A similar argument shows the estimate

$$|N^h(u) - N^h(v)|_{h,(1/2)} \leqslant Cm\, \Delta t\, |u - v|_{h,(1/2)}.$$

Now we estimate L^h. From the spectral properties of L^h we have

$$|L^h x|_\gamma \leqslant (1 + \Delta t A^h)^{-m}\, |x|_\gamma \qquad \forall x \in Z^h,$$

and

$$(1+\Delta t\lambda^h)^{-m}\,|x|_\gamma \leqslant |L^h x|_\gamma \leqslant (1+\Delta\lambda_1^h)^{-m}\,|x|_\gamma \qquad \forall x\in Y^h.$$

Without loss of generality we may choose T such that $T/\Delta t$ is an integer. If we fix m such that $m\Delta t = T$, we have

$$|L^h x|_\gamma \leqslant e^{-\tilde{\Lambda}T}|x|_\gamma \qquad \forall x\in Z^h,$$

and

$$e^{-\tilde{\lambda}T}|x|_\gamma \leqslant |L^h x|_\gamma \leqslant e^{-\tilde{\lambda}_1 T}|x|_\gamma \qquad \forall x\in Y^h,$$

where

$$\tilde{\Lambda}=\frac{\ln(1+\Delta t\Lambda^h)}{\Delta t}, \qquad \tilde{\lambda}=\frac{\ln(1+\Delta t\lambda^h)}{\Delta t}, \qquad \tilde{\lambda}_1=\frac{\ln(1+\Delta t\lambda^1)}{\Delta t}.$$

Hence, $\tilde{\Lambda}\to\Lambda^0$, $\tilde{\lambda}\to\lambda^0$, $\tilde{\lambda}_1\to\lambda_1^0$ as h, $\Delta t\to 0$ under (6.14).

Thus by following the proof of Lemma 4.1 it follows that $(G^h1)-(G^h3)$ are satisfied for G^h defined by $G^h(u)=L^h(u)+N^h(u)$, $m=T/\Delta t$. With the exception of (G^h5), $(G^h4)-(G^h7)$ all follow as in the proof of Theorem 6.4. (G^h5) requires an error estimate for the prepared equations where as the proof in [47] is for the unprepared equations. The difference is that the only Lipschitz property available for the prepared nonlinear term $F(u)$ is $|F(u)-F(v)|\leqslant K\,|u-v|_\gamma$, while [47] also uses $|f(u)-f(v)|_{-1}\leqslant K\,|u-v|$. This problem also occurred in Section 5 for the spectral method and can be overcome in the same way. Indeed, the proof follows the proof of (G^h5) in Theorem 5.3 (which follows [47]) only here we use the estimate $|A^\varepsilon u^h|\leqslant h^{2\varepsilon}K\,|u^h|$ for all $u^h\in V^h$ (as opposed to $|A^\varepsilon P^N u|\leqslant\lambda_N^\varepsilon\,|P^N u|$ in the spectral case).

The Conditions C' also follow as in Theorem 6.4, where now we set

$$\Lambda=\inf_{\substack{0\leqslant h\leqslant h_0\\ \Delta t=dh}}\tilde{\Lambda}, \qquad \lambda=\sup_{\substack{0\leqslant h\leqslant h_0\\ \Delta t=dh}}\tilde{\lambda}, \qquad \lambda_1=\inf_{\substack{0\leqslant h\leqslant h_0\\ \Delta t=dh}}\tilde{\lambda}_1.$$

Hence an invariant manifold $\mathcal{M}^{h,\Delta t}$ has been constructed. Finally we remark that $\mathcal{M}^{h,\Delta t}$ is an invariant manifold for the discrete semigroup $S^{h,\Delta t}(m)$. An argument similar to that used to show that $\mathcal{M}$ is invariant for $S(t)$, for all $t>0$, in Theorem 4.2, shows that the $\mathcal{M}^{h,\Delta t}$ is also invariant for $S^{h,\Delta t}(1)$. ∎

7. Appendix

In this section we supply the missing estimates needed in the proofs of Section 5. We suppose throughout this section that $u(t)$ is the solution of

(4.3) and that Assumptions E of Section 4 hold along with (5.9). We also require $0 \leqslant \beta < 1$. Let us begin with a lemma whose utility will become clear in the subsequent lemmas.

LEMMA 7.1. *Let $u(t)$ solve (4.3) with $u_0 \in X$. Then for $0 < \tau < s < t \leqslant T$ we have*

$$|u(t)-u(s)|_\gamma \leqslant \frac{K_\alpha (t-s)^\alpha}{(s-\tau)^\alpha} |u(\tau)|_\gamma + \frac{K(t-s)}{(s-\tau)^\beta}$$

for all $0 < \alpha \leqslant 1$.

Proof. We have from (5.3)

$$u(t)-u(s) = (e^{-(t-s)A} - I)\, e^{-(s-\tau)A} u(\tau) + \int_\tau^s e^{-(t-r)A}$$

$$\times (F(u(r+t-s)) - F(u(r)))\, dr + \int_\tau^{\tau+t-s} e^{-(t-r)A} F(u(r))\, dr.$$

Using the estimates $|e^{-tA}u - u| \leqslant b_\alpha t^\alpha |u|_\alpha$, $0 < \alpha \leqslant 1$ (see [52] p. 74), $|A^\beta e^{-tA}| \leqslant K t^{-\beta}$ and assumption ($E2$), we obtain

$$|u(t)-u(s)|_\gamma \leqslant \frac{b_\alpha (t-s)^\alpha}{(s-\tau)^\alpha} |u(\tau)|_\gamma + \frac{K(t-s)}{(s-\tau)^\beta}$$

$$+ \int_\tau^s \frac{K}{(s-r)^\beta} |u(r+t-s) - u(r)|_\gamma\, dr.$$

The result follows after an application of the Henry-Gronwall lemma [31]. ∎

We may now obtain a bound on $|du/dt|_{\gamma-\beta}$. This in turn will allow us to estimate $|u(t)|_{1+\gamma-\beta}$ on the interval $(0, T]$. To accomplish this we modify Lemma 3.5.1 of Henry [31] (see also [52] p.114) which essentially says that if $g\colon (0, T) \mapsto X$ is such that $|g(t)-g(s)| \leqslant K(s)|t-s|^\eta$ for $0 < \eta \leqslant 1$, $0 < s < t < T$ and $K(s) \in L^1((0, T))$, then $G(t) := \int_0^t e^{-(t-s)A} g(s)\, ds$ is continuously differentiable on $(0, T)$.

LEMMA 7.2. *There exists a constant $K = K(T)$ such that*

$$\left|\frac{du}{dt}(t)\right|_{\gamma-\beta} \leqslant \frac{K}{t^{1-\beta}}, \qquad \forall t \in (0, T],$$

provided $u_0 \in \mathscr{D}(A^\gamma)$.

Proof. Since $u(t)$ is Hölder continuous, so is $F(u(t))$. Thus by Lemma 3.5.1 of Henry, [31] (5.3) is differentiable and

$$\frac{du}{dt}(t) = -Ae^{-tA}u(\tau) + e^{-(t-\tau)A}F(u(t))$$

$$+\int_\tau^t Ae^{-(t-s)A}(F(u(t)) - F(u(s)))\,ds.$$

We then have

$$\left|\frac{du}{dt}(t)\right|_{\gamma-\beta} \leqslant \frac{K}{t^{1-\beta}}|u(\tau)|_\gamma + K_1 + \int_\tau^t \frac{K}{t-s}|u(t)-u(s)|_\gamma\,ds.$$

The result follows after using the previous lemma and the fact that since $u_0 \in \mathscr{D}(A^\gamma)$, $|u(\tau)|_\gamma$ remains bounded as $\tau \to 0$. ∎

Since $Au = -du/dt - F(u(t))$, we have

COROLLARY 7.3. *Under the above assumptions there exists a constant $K = K(T)$ such that*

$$|u(t)|_{1+\gamma-\beta} \leqslant \frac{K}{t^{1-\beta}}, \qquad \forall t \in (0, T].$$

Now that we have $|u(t)|_{1+\gamma-\beta}$ is bounded uniformly on $0 < \tau \leqslant t \leqslant T$ we may improve Lemma 7.1.

COROLLARY 7.4. *Let $u(t)$ solve (4.3) with $u_0 \in \mathscr{B}_\gamma(0, \rho)$. Then for $0 < s < t \leqslant T$ and given any $\varepsilon > 0$ such that $\beta + \varepsilon < 1$ and $0 < \alpha < 1$, we have*

$$\left|\frac{du}{dt}(t) - \frac{du}{dt}(s)\right|_{\gamma-1+\varepsilon} \leqslant K_\varepsilon\left(\frac{(t-s)^\alpha}{s^{\alpha+\varepsilon}} + \frac{(t-s)}{s^{\beta+\varepsilon}}\right).$$

Proof. Since $u_0 \in \mathscr{D}(A^\gamma)$, we may set $\tau = 0$ in Lemma 7.1. The proof follows Lemma 7.1. We have

$$|u(t) - u(s)|_{\gamma+\varepsilon} \leqslant \frac{b_\alpha(t-s)^\alpha}{s^{\alpha+\varepsilon}}|u(0)|_\gamma + \frac{K(t-s)}{s^{\beta+\varepsilon}}$$

$$+\int_0^s \frac{K}{(s-r)^{\beta+\varepsilon}}|u(r+t-s) - u(r)|_\gamma\,dr.$$

Now we use Lemma 7.1 and our assumptions on β, ε to conclude

$$|u(t)-u(s)|_{\gamma+\varepsilon} \leqslant K_\varepsilon \left(\frac{(t-s)^\alpha}{s^{\alpha+\varepsilon}} + \frac{(t-s)}{s^{\beta+\varepsilon}}\right).$$

The result follows since

$$A^{\gamma-1+\varepsilon}\frac{du}{dt} = -A^{\gamma+\varepsilon}u(t) - A^{\gamma-1+\varepsilon}F(u(t))$$

and $\gamma - 1 + \varepsilon \leqslant \gamma - \beta$. ∎

Now we turn to the estimates on u_{tt}. Set $v(t) = t^2 u_t(t)$. Formally v satisfies

$$\begin{aligned} \frac{dv}{dt} + Av &= 2t\frac{du}{dt} - F'(u(t))\, v, \\ v(0) &= 0. \end{aligned} \tag{7.1}$$

A mild solution of this equation is of the form

$$v(t) = \int_0^t e^{-(t-s)A}\left(2s\frac{du}{ds}(s) - F'(u(s))\, v(s)\right) ds. \tag{7.2}$$

One can show that the map H defined on the complete metric space $C([0, \tau], X)$ by

$$Hw = \int_0^\tau A^\gamma e^{-(t-s)A}\left(2s\frac{du}{ds}(s) - F'(u(s))\, A^{-\gamma}w(s)\right) ds$$

has a fixed point for τ sufficiently small. Hence $A^\gamma v(t) = w(t)$ is bounded in X for $0 \leqslant t \leqslant \tau$ where $v(t)$ solves (7.2) (see [31] for example).

Moreover, we have from (7.2) and Lemma 7.2

$$\begin{aligned} |v(t)|_\gamma &\leqslant K\int_0^t \frac{s^\beta}{(t-s)^\beta} + K\int_0^t \frac{|v(s)|_\gamma}{(t-s)^\beta} \\ &\leqslant Kt + K\int_0^t \frac{|v(s)|_\gamma}{(t-s)^\beta}. \end{aligned}$$

After an application of the Henry-Gronwall lemma, [31], we obtain

$$|v(t)|_\gamma \leqslant Kt \qquad 0 \leqslant t \leqslant T. \tag{7.3}$$

This shows that the maximal interval of existence of the solution of (7.2) is $0 \leqslant t \leqslant T$. More generally one can show

$$|v(t)|_{\gamma-\varepsilon} \leqslant Kt^{1+\varepsilon}. \tag{7.4}$$

LEMMA 7.5. *Let $u(t)$ solve (4.3) with $u_0 \in \mathscr{B}_\gamma(0, \rho)$. There exists a positive constant $K = K(\eta, T)$ such that*

$$|u_{tt}(t)|_{-\eta} \leqslant \frac{K}{t^{2-\eta-\gamma}}, \qquad \forall t \in (0, T] \tag{7.5}$$

holds for all $\eta > -\gamma$.

Proof. As in Lemma 7.2 v is continuously differentiable provided we can show $v(t)$ is Hölder continuous in $\mathscr{D}(A^\gamma)$ for $0 < t \leqslant T$. To accomplish this we proceed as in Lemma 7.1. With the help of Corollary 7.4 and considerable computations we find

$$\begin{aligned} |v(t) - v(s)|_\gamma \leqslant K[s^{-\beta}(t-s)^{1+\beta} &+ (1 + ts^{1-2\beta})(t-s) \\ &+ (1 + ts^{1-\alpha-\beta})(t-s)^\alpha]. \end{aligned} \tag{7.6}$$

Thus $v(t)$ is continuously differentiable, and we have

$$\begin{aligned} \left|\frac{dv}{dt}(t)\right|_{-\eta} = \Bigg| & A^{-\eta}e^{-At}\left(2t\frac{du}{dt} - F'(u(t))\,v(t)\right) \\ & + \int_0^t A^{-\eta}e^{-(t-s)A}\left(2t\frac{du}{dt}(t) - 2s\frac{du}{dt}(s)\right) ds \\ & - \int_0^t A^{-\eta}e^{-(t-s)A}(F'(u(t))\,v(t) - F'(u(s))\,v(s))\,ds\Bigg|. \end{aligned}$$

Using Corollary 7.4, (7.6) , (7.3), Lemma 7.2, and the estimate $|A^\eta e^{-At}| \leqslant Kt^{-\eta}$, we arrive at

$$\left|\frac{dv}{dt}(t)\right|_{-\eta} \leqslant Kt^{\eta+\gamma}.$$

Repeating the analysis of Lemma 7.2 under the same assumptions on β, γ, η, we find

$$\left|\frac{du}{dt}(t)\right|_{-\eta} \leqslant Kt^{\gamma+\eta-1}.$$

The result follows after noticing $v_t = 2tu_t + t^2u_{tt}$. ∎

We conclude with the following lemma which is the discrete analog of the estimate $|A^{\alpha}e^{-At}| \leqslant c_{\alpha}t^{-\alpha}$.

LEMMA 7.6. *Let $\alpha \leqslant 1$ be given. There exists a constant c_{α} which remains bounded as $\alpha \to 1$ provided $n \neq 1$ such that*

$$|A^{\alpha}(I + \Delta t A)^{-n}| \leqslant \frac{c_{\alpha}}{(\Delta t n)^{\alpha}}$$

for all $n > 1$.

Proof. The bound follows from noticing that the maximum of the function $f(x) = x^{\alpha}(1 + \Delta t x)^{-n}$ occurs at $x = \alpha/(\Delta t(n - \alpha))$. ∎

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