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# Existence of Solutions of a Two-Point Free-Boundary Problem Arising in the Theory of Porous Medium Combustion

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The existence of solutions of a two-point free-boundary problem arising from the theory of travelling combustion waves in a porous medium is examined. The problem comprises a third-order nonlinear ordinary differential equation posed on an unknown interval of finite length; four boundary conditions are given, two at either end of the interval. The equations possess a trivial solution for all values of the bifurcation parameter $\lambda$. A shooting technique is employed to prove the existence of a nontrivial solution for $0<\lambda<\lambda_c$ and nonexistence theorems are proved for $\lambda \notin (0, \lambda_c)$.



## 1. Introduction

IN this paper we examine a two-point free-boundary problem (FBP) arising in the study of travelling combustion waves in a porous medium. The equations are derived in Section 5 of [2] and discussed further in [3]. They represent the leading-order nonlinear eigenvalue problem found in a series expansion of the equations governing the existence of steadily propagating combustion waves for small driving velocities. We describe the derivation of the equations at the end of this section.

The equations possess a single trivial solution, and the objective of this paper is to establish the existence of at least one nontrivial solution of the problem within a certain parameter regime. We employ a shooting technique and establish the existence of an odd number (greater than or equal to one) of nontrivial solutions in the required parameter range. Outside this range we prove that nontrivial solutions do not exist.

In Section 2 we define the free-boundary problem and in Section 3 we re-formulate it as a shooting problem. Section 4 contains nonexistence results for certain ranges of values of the distinguished parameter. In Section 5 we establish some preliminary results on the behaviour of solutions of the initial-value problem that defines the shooting problem and finally, in Section 6, we prove the existence theorem.

We briefly describe the origin of the free-boundary problem and its physical interpretation. When seeking planar travelling-wave solutions in the model for porous medium combustion described in [2], the following fourth-order ordinary differential equation is derived; we denote by $PC^n$ the space of functions that are piecewise $C^n$ on the whole real line, and employ the notation $' \equiv d/dx$. The problem may be stated as:

find $(U, W, Q, c) \in \mathrm{PC}^2 \times \mathrm{PC}^3 \times \mathrm{PC}^1 \times \mathbb{R}$ satisfying

$$U'' + QU' + W - U + r = 0, \qquad \mu W' = U - W, \qquad Q' = \lambda r,$$

where
$$r = \mathrm{H}(U - \eta)\mathrm{H}(Q)\mu^{\frac{1}{2}} f(W), \tag{1.1}$$

together with the boundary conditions

$$\lim_{x\to\infty} U(\pm x) = \lim_{x\to\infty} W(-x) = u_a < \eta, \qquad \lim_{x\to\infty} Q(x) = c. \tag{1.2}$$

Here, $\mathrm{H}(\bullet)$ is the Heaviside unit step function and the function $f(W) \in \mathrm{C}^1(U_a, \infty)$. Solutions of the problem form a heteroclinic orbit in the four-dimensional phase space of the problem.

The variable $U(x)$ represents the temperature of the solid phase while $W(x)$ represents the temperature of the gaseous phase. The eigenvalue $c$, which appears only in the boundary condition (1.2), is the speed of propagation of the combustion wave. $Q(x)$ represents the product of the wave speed $c$ and the (variable) heat capacity of the solid medium; the heat capacity is a linear function of the concentration of combustible solid and so determination of $Q(x)$ and $c$ determines the profile of the solid reactant. Equation (1.1) defines the reaction rate $r$. The function $f(W)$ has been shown experimentally to be of the form $f(W) = W^2$. The distinguished parameters are $\mu$ and $\lambda$ which represent, respectively, the scaled inlet gas velocity (which drives the combustion process) and a linear function of the specific heat of the reactant.

For $r = 0$ the governing ordinary differential equations are linear and explicitly solvable. Thus, by integrating the equations in the regions where $r = 0$, and by imposing suitable continuity conditions on the components of the solution, a free-boundary problem may be derived [2]. The unknown interval on which the free-boundary problem is posed determines the extent in $x$-space in which the exothermic chemical reaction occurs ($r \neq 0$).

We examine solutions of the free-boundary problem in the limit as $\mu \to 0$. Numerical studies [2] indicate that, for $\mu < 1$, a nontrivial travelling-wave solution exists for all $\lambda \in (0, \lambda_c)$, where $\lambda_c = (\eta - u_a)^{-1}$. In order to go some way towards verifying this numerical observation, we prove the result for $\mu \ll 1$. We define $L$ to be the greatest number such that $r \neq 0$ for $x \in (0, L)$. Then, as $\mu \to 0$, a dimensional analysis shows that

$$U \sim U_0, \qquad W \sim W_0, \qquad Q \sim \mu^{\frac{1}{4}} Q_0, \qquad L \sim \mu^{-\frac{1}{4}} L_0, \tag{1.3}$$

where the subscript zero denotes an order-one quantity (with respect to $\mu$). Thus, the free-boundary problem considered in the remainder of this paper is the leading-order problem for $U_0$, $W_0$, $Q_0$, and $L_0$. For notational convenience the subscripts zero are dropped henceforth. Note that $W_0 = U_0$ so that $W_0$ does not appear explicitly.

## 2. The governing equations

As in Section 1, we employ the notation $' \equiv \mathrm{d}/\mathrm{d}x$. The free-boundary problem may be stated as follows:

find $(Q, U, L) \in C^1(0, L) \times C^2(0, L) \times \mathbb{R}^+$, where

$$Q' = \lambda f(U), \tag{2.1}$$

$$U'' + QU' + f(U) = 0, \tag{2.2}$$

subject to the boundary conditions

$$Q(0) = 0, \qquad U'(0) = 0, \qquad U(L) = \eta, \qquad \lambda_c U'(L) + Q(L) = 0. \tag{2.3}$$

In addition, we seek solutions satisfying $U(x) - \eta \geq 0$ for $x \in [0, L]$. This condition is a consequence of the form of the reaction rate (1.1) in the original problem. The parameter $\lambda_c$ may be any real, while $\eta \in \mathbb{R}^+$. In general, we shall consider $f(U)$ to be a strictly positive $C^1(\eta, \infty)$ function with strictly positive first derivative. In the proof of Lemmas 5.3–5.5, where more specific properties of the function are required, we will take $f(U) = U^2$, since this case arises in the practical application of porous medium combustion [2].

## 3. Formulation as a shooting problem

We may reformulate the free-boundary problem defined by (2.1)–(2.3) as a shooting problem, namely to locate the zeros of the parameter-dependent functional $G(\xi)$, where

$$G(\xi) = \lambda_c U'(L; \xi) + Q(L; \xi). \tag{3.1}$$

Here, $U(x; \xi)$ and $Q(x; \xi)$ are the solutions of equations (2.1)–(2.2) subject to the initial conditions

$$U(0; \xi) = \xi, \qquad U'(0; \xi) = 0, \qquad Q(0; \xi) = 0. \tag{3.2}$$

$L(\xi)$ is defined to be the first zero of $U(L; \xi) - \eta$, so that

$$U(L; \xi) = \eta. \tag{3.3}$$

Since we are interested in solutions for which $U(x) - \eta \geq 0$ for all $x \in [0, L]$, we require that $\xi \geq \eta$.

NOTES (i) The shooting problem defined by (3.1) possesses a trivial solution $\xi = \eta$. This corresponds to the trivial solution of equations (2.1)–(2.3), given by

$$Q \equiv 0, \qquad U \equiv \eta, \qquad L = 0. \tag{3.4}$$

(ii) It is proved in [2] that a branch of nontrivial solutions of the free-boundary problem bifurcates subcritically from the trivial solution at $\lambda = \lambda_c$. In Theorem 6.1, we extend this result to a global existence theorem for $0 < \lambda < \lambda_c$.

(iii) Henceforth, for ease of notation, we will denote the functions $U(x; \xi)$ and $Q(x; \xi)$ by $U(x)$ and $Q(x)$.

In order that the functional defined by equation (3.1) is well defined, it is necessary that equation (3.3) determines a finite value of $L$ for each finite $\xi \geq \eta$. This result is now proved as a corollary of the following lemma..

LEMMA 3.1 *For $\xi > U(x) \geq \eta$, the function $U(x)$ is strictly monotone decreasing.*

*Proof.* From (2.2) it is clear that, at any point where $U'=0$, we have $U''=-f(U)$. Since $f(U)$ is strictly positive for $U \geqslant \eta$, we deduce that $U$ cannot attain a minimum for $U \geqslant \eta$, nor can it tend to a limiting value $U(\infty) \geqslant \eta$. Thus, since $U'(0)=0$ (by (2.3)), $U$ is a strictly monotone decreasing function for $\xi > U \geqslant \eta$. □

COROLLARY 3.2 *For each finite $\xi \geqslant \eta$, there exists a value $x=L$ such that the condition* (3.3) *is satisfied.*

*Proof.* The result of Lemma 3.1 implies that $U(x)$ must reach $\eta$ at a finite value of $x$, since $U(x)$ cannot oscillate above $\eta$, blow up, or tend to a finite limit $\geqslant \eta$. Consequently, $L$ is well defined for all $\xi \geqslant \eta$. □

## 4. Nonexistence theorems

In this section, we prove that solutions of the shooting problem defined by (3.1) cannot exist outside a certain parameter range of $\lambda$.

THEOREM 4.1 *Nontrivial solutions of the shooting problem defined by* (3.1) *do not exist for $\lambda \leqslant 0$ and for $\lambda \geqslant \lambda_c$.*

*Proof.* ($\lambda \leqslant 0$) Since $f(U)$ is positive for $U \geqslant \eta$, we deduce from (2.2)–(2.3) that $Q(x) \leqslant 0$ for $0 \leqslant x \leqslant L$. Also, by Lemma 3.1, we know that $U'(x)<0$ for $0<x \leqslant L$. Thus $G(\xi)<0$.

($\lambda \geqslant \lambda_c$) Combining equations (2.1)–(2.2) to eliminate $f(U)$ and integrating with respect to $x$ gives us

$$Q(L)+\lambda U'(L) = -\int_0^L \lambda Q U' \, dx.$$

But, since $U'(x)<0$ for $0<x<L$ (by Lemma 3.1), and $Q(x)>0$ for $0<x<L$, we have

$$G(\xi)=Q(L)+\lambda_c U'(L) \geqslant Q(L)+\lambda U'(L)>0.$$

Thus (3.1) cannot be satisfied and the proof is complete. □

## 5. Preliminary results

In this section, we prove various results needed to prove the existence theorem in Section 6. Lemmas 5.1 and 5.2 apply to the general function $f(U)$. The remaining lemmas in the section, however, apply to the specific case $f(U)=U^2$.

LEMMA 5.1

$$\lim_{\xi \to \eta_+} G_\xi(\xi) = \operatorname{sgn}(\lambda-\lambda_c)\infty.$$

*Proof.* By differentiating equation (3.1) with respect to $\xi$, we obtain

$$G_\xi(\xi)=\lambda_c U'_\xi(L)+Q_\xi(L)+[\lambda_c U''(L)+Q'(L)]L_\xi. \tag{5.1}$$

Here $U_\xi$ and $Q_\xi$ satisfy the initial-value problem

$$Q'_\xi = \lambda f_U(U)U_\xi, \qquad U''_\xi + Q_\xi U' + QU'_\xi + f_U(U)U_\xi = 0,$$

subject to

$$Q_\xi(0) = 0, \qquad U'_\xi(0) = 0, \qquad U_\xi(0) = 1. \tag{5.2}$$

Differentiating the expression (3.3) with respect to $\xi$ we obtain the following expression to determine $L_\xi$:

$$U'(L)L_\xi + U_\xi(L) = 0. \tag{5.3}$$

Using (2.2) and (5.3) to eliminate $U''(L)$ and $L_\xi$, respectively, from (5.1), we obtain

$$G_\xi(\xi) = \lambda_c U'_\xi(L) + Q_\xi(L) + \lambda_c Q(L)U_\xi(L) - (\lambda - \lambda_c)f(U(L))U_\xi(L)/U'(L). \tag{5.4}$$

However, as $\xi \to \eta$, the solution of the initial-value problem defining (3.1) approaches the trivial solution defined by (3.4). Thus (5.4) yields

$$\lim_{\xi\to\eta_+} G_\xi(\xi) = \lambda_c U'_\xi(0) + Q_\xi(0) + \lambda_c Q(0)U_\xi(0) - \lim_{\xi\to\eta_+} \frac{(\lambda-\lambda_c)f(U(L))U_\xi(L)}{U'(L)}.$$

Applying the initial conditions (3.2) and (5.2) and noting that

$$\lim_{\xi\to\eta_+} U'(L) = \lim_{L\to 0_+} U'(L) = 0_-$$

(since $U'(L)$ is necessarily negative by Lemma 3.1), we obtain

$$\lim_{\xi\to\eta_+} G_\xi(\xi) = \operatorname{sgn}(\lambda - \lambda_c)\infty. \quad \square$$

LEMMA 5.2 *For $\lambda > 0$, there exists at most one point $x = s$ in the interval* $(0, L]$ *at which $U''(s) = 0$.*

*Proof.* Differentiate equation (2.2) with respect to $x$ and eliminate $Q'(x)$ by using equation (2.1). This yields

$$U''' + \lambda f(U)U' + QU'' + f_U(U)U' = 0.$$

Since $f(U)$ and $f_U(U)$ are both strictly positive and $U'(x) < 0$ for $\eta \leq U \leq \infty$, we deduce that at any point where $U'' = 0$ we have $U''' > 0$. Thus $U''$ can equal zero at most once in $(0, L]$. $\square$

LEMMA 5.3 *For $\lambda > 0$ and $f(U) = U^2$, the solution of the initial-value problem defined by equations* (2.1)–(2.2) *and* (3.2) *satisfies*

$$U \geq \xi \exp(-x/2\lambda^{\frac{1}{2}}), \qquad Q \geq \tfrac{1}{2}\lambda^{\frac{1}{2}}\xi^2[1 - \exp(-2x/\lambda^{\frac{1}{2}})],$$

*for $0 \leq x \leq s$, where $s$ is as defined in Lemma* 5.2. *If such an $s$ does not exist then we replace $s$ by $L$.*

*Proof.* Eliminating $f(U)$ between equations (2.1) and (2.2) and integrating yields

$$U'(x) + Q(x)/\lambda + \int_0^x (QU')(y)\,dy = 0. \tag{5.5}$$

Eliminating $Q(x)$ from (2.2) by using (5.5) gives us

$$U'' - \lambda U'^2 + U^2 = \lambda\left(\int_0^x (QU')(y)\,dy\right)U'.$$

Since $Q(x) \geq 0$, $U'(x) \leq 0$, and $U''(x) \leq 0$ for $0 \leq x \leq s$, we deduce that

$$\lambda U'^2 \leq U^2 \quad \text{for} \quad 0 \leq x \leq s.$$

Thus

$$U'/U \geq -1/\lambda^{\frac{1}{2}}. \tag{5.6}$$

Integrating this differential inequality and applying the initial condition (3.2) on $U(0)$, we obtain

$$U(x) \geq \xi \exp(-x/\lambda^{\frac{1}{2}}). \tag{5.7}$$

Applying this inequality to equation (2.1) gives us

$$Q(x) \geq \tfrac{1}{2}\lambda^{\frac{1}{2}}\xi^2[1 - \exp(-2x/\lambda^{\frac{1}{2}})]. \tag{5.8}$$

This completes the proof. □

LEMMA 5.4 *For $\lambda > 0$ and $f(U) = U^2$, the point $s$ defined in Lemma 5.2 satisfies*

$$s \geq (2\xi\lambda^{\frac{1}{2}})^{-1} \quad \text{for} \quad \xi > 2\lambda^{-1}. \tag{5.9}$$

*Proof.* The point $s$ is defined by $U''(s) = 0$. Since $U(x) \leq \xi$ for $x \geq 0$, equations (2.1) and (5.6) give us $Q(x) \leq \lambda\xi^2 x$ and $U'(x) \geq -\xi/\lambda^{\frac{1}{2}}$ for $0 \leq x \leq s$. Substituting these inequalities and the bound (5.7) into (2.2) we deduce that, for $0 \leq x \leq s$,

$$-U''(x) = (QU')(x) + U^2(x) \geq -\lambda^{\frac{1}{2}}\xi^3 x + \xi^2 \exp(-2x/\lambda^{\frac{1}{2}}).$$

This gives $s \exp(2s/\lambda^{\frac{1}{2}}) \geq 1/\xi\lambda^{\frac{1}{2}}$. Hence $s \geq s_1$, where

$$F(s_1) = s_1 \exp(2s_1/\lambda^{\frac{1}{2}}) - 1/\xi\lambda^{\frac{1}{2}} = 0,$$

since $F(\cdot)$ is a monotonically increasing function of its argument. Also $F(0) < 0$ and $F(1/\xi\lambda^{\frac{1}{2}}) > 0$. Thus, by continuity,

$$0 < s_1 < 1/\xi\lambda^{\frac{1}{2}}.$$

By choosing $\xi > 2\lambda^{-1}$ we obtain $2s_1/\lambda^{\frac{1}{2}} < 1$. Thus, by inequality (4.2.31) in [1], we have

$$0 = F(s_1) \leq s_1(1 - 2s_1/\lambda^{\frac{1}{2}})^{-1} - 1/\xi\lambda^{\frac{1}{2}}.$$

Solving this inequality yields

$$s \geq s_1 \geq (\xi\lambda^{\frac{1}{2}} + 2\lambda^{-\frac{1}{2}})^{-1} \geq (2\xi\lambda^{\frac{1}{2}})^{-1}$$

since $\xi > 2\lambda^{-1}$.

LEMMA 5.5 *For $\lambda > 0$ and $f(U) = U^2$, we have*

$$\lim_{\xi\to\infty} L(\xi) = \infty.$$

*Proof.* By virtue of Lemma 5.4 we need only consider the following three cases:

(i) $1/2\xi\lambda^{\frac{1}{2}} < s < a$, where $a$ is bounded above independently of $\xi$;

(ii) $s \to \infty$ as $\xi \to \infty$ or $\nexists\, s < L : U''(s) = 0$;

(iii) $s$ oscillates unboundedly as $\xi \to \infty$ but contains neighbourhoods in which it is bounded.

CASE (i) Consider $\xi > 2\lambda^{-1}$. At $x = s$ we have $U''(s) = 0$. Thus, equation (2.2) implies that

$$U'(s) = -U^2(s)/Q(s) \geq -2/\lambda^{\frac{3}{2}}[1 - \exp(-2s/\lambda^{\frac{1}{2}})]$$

since $U(x) \leq \xi$ for $x \geq 0$ and since (5.8) holds for $0 \leq x \leq s$. By Lemma 5.4 and inequality (4.2.32) in [1], we have, for $\xi > 2\lambda^{-1}$,

$$1 - \exp(-2s/\lambda^{\frac{1}{2}}) > 1 - \exp(-1/\xi\lambda) > (\xi\lambda + 1)^{-1} > \tfrac{2}{3}(\xi\lambda)^{-1}. \tag{5.10}$$

Substituting this into the inequality for $U'(s)$, we obtain

$$U'(s) \geq -3\xi/\lambda^{\frac{1}{2}}. \tag{5.11}$$

Thus, since (by Lemma 5.2) $U''(x) \geq 0$ for $x \geq s$, we have

$$U(x) \geq U(s) - 3\xi(x - s)/\lambda^{\frac{1}{2}} \quad \text{for} \quad x \geq s.$$

By (5.7), since $s < a$, we deduce that

$$U(x) \geq \xi b - 3\xi(x - s)/\lambda^{\frac{1}{2}} \quad \text{for} \quad x \geq s, \tag{5.12a}$$

where

$$b = \exp(-a/\lambda^{\frac{1}{2}}) \tag{5.12b}$$

is uniformly bounded above with respect to $\xi$.

We define the point $x = s_2$ by $U(s_2) = \xi^{\frac{1}{2}}$. Then, by (5.12),

$$3(s_2 - s) \geq \lambda^{\frac{1}{2}}b - (\lambda/\xi)^{\frac{1}{2}}. \tag{5.13}$$

Substituting expression (5.12a) for $U(x)$ into equation (2.1) and integrating, we obtain, for $0 \leq x \leq s$,

$$Q(x) \geq Q(s) - \tfrac{1}{9}\lambda^{\frac{1}{2}}\xi^2[b - 3(x - s)/\lambda^{\frac{1}{2}}]^3 + \tfrac{1}{9}\lambda^{\frac{1}{2}}\xi^2 b^3.$$

Applying (5.12a) gives us $Q(s_2) \geq \frac{1}{9}\lambda^{\frac{1}{2}}\xi^2 b^3 + Q(s) - \frac{1}{9}\lambda^{\frac{1}{2}}\xi^{\frac{1}{2}}$. Inequalities (5.8) and (5.10) imply that $Q(s) \geq \frac{1}{3}\lambda^{\frac{1}{2}}\xi$. Thus, for $\xi \geq \frac{1}{9}\lambda^2$, we deduce that

$$Q(s_2) \geq c\xi^2 = \tfrac{1}{9}\lambda^{\frac{1}{2}}\xi^2 b^2. \tag{5.14}$$

Since $Q(x)$ is an increasing function and $U'(x) < 0$, we have, for $x > s_2$,

$$U'' + c\xi^2 U' \geq U'' + QU' = -U^2 \geq -\xi.$$

From this we obtain $[\exp(c\xi^2 x)U']' \geq = -\xi \exp(c\xi^2 x)$. Integrating, and evaluating the constant of integration at $x = s_2$, gives us

$$U' \geqslant \exp[c\xi^2(s_2 - x)] \cdot U'(s_2) + \{\exp[c\xi^2(s_2 - x)] - 1\}/c\xi.$$

Integrating again implies that, for $x \geqslant s_2$,

$$U \geqslant \xi^{\frac{1}{2}} - \frac{\exp[c\xi^2(s_2 - x)]U'(s_2)}{c\xi^2} + \frac{U'(s_2)}{c\xi^2} - \frac{\exp[c\xi^2(s_2 - x)]}{c^2\xi^3} - \frac{x}{c\xi} + \frac{1}{c^2\xi^3} + \frac{s_2}{c\xi}.$$

Thus, in particular, since $U(L) = \eta$ from (3.3), we have

$$\frac{L}{c\xi} \geqslant \xi^{\frac{1}{2}} - \frac{\exp[c\xi^2(s_2 - L)]U'(s_2)}{c\xi^2} + \frac{U'(s_2)}{c\xi^2} - \frac{\exp[c\xi^2(s_2 - L)]}{c^2\xi^3} + \frac{1}{c^2\xi^3} + \frac{s_2}{c\xi} - \eta.$$

Since $L > s_2 > 0$ and $U'(s_2) < 0$, we obtain

$$L/c\xi \geqslant \xi^{\frac{1}{2}} + U'(s_2)/c\xi^2 - \eta.$$

Finally, since $U'' > 0$ for $x > s$ (by Lemma 5.2), we deduce from (5.11) that $U'(s_2) \geqslant U'(s) \geqslant -3\xi/\lambda^{\frac{1}{2}}$. Thus

$$L/c\xi \geqslant \xi^{\frac{1}{2}} - 3/c\xi\lambda^{\frac{1}{2}} - \eta \tag{5.15}$$

for $\xi \geqslant \max\{2\lambda^{-1}, \frac{1}{9}\lambda^2\}$. This implies that $L \to \infty$ as $\xi \to \infty$, since, by (5.12b) and (5.14), we have

$$c = \tfrac{1}{9}\lambda^{\frac{3}{2}}b^3 = \tfrac{1}{9}\lambda^{\frac{3}{2}}\exp(-3a/\lambda^{\frac{1}{2}}), \tag{5.16}$$

where $a$ is bounded above independently of $\xi$.

CASE (ii) If $L > s$ for $\xi$ sufficiently large, then, since $s \to \infty$ as $\xi \to \infty$, we deduce that $L \to \infty$ as $\xi \to \infty$.

If $L < s$ for $\xi$ sufficiently large (noting that $s$ may be infinite), then Lemma 5.3 gives us $\eta \geqslant \xi \exp(-L/2\lambda^{\frac{1}{2}})$ and hence that

$$L \geqslant 2\lambda^{\frac{1}{2}} \ln(\xi/\eta).$$

Thus $L \to \infty$ as $\xi \to \infty$.

Finally, if $L$ oscillates about $s$ for $\xi$ sufficiently large, we obtain

$$L \geqslant L_1 = \min\{2\lambda^{\frac{1}{2}} \ln(\xi/\eta), s\} \tag{5.17}$$

and thus we have the desired result.

CASE (iii) We subdivide this case:

*Part (a)* The unbounded oscillations of $s$ satisfy

$$s < \tfrac{1}{3}\lambda^{\frac{1}{2}} \ln \xi \quad \text{as} \quad \xi \to \infty. \tag{5.18}$$

In this part we modify the proof of case (i) to allow $a = \frac{1}{3}\lambda^{\frac{1}{2}} \ln \xi$. Substituting this into the expression (5.15) for $L$, where $c$ is defined by (5.16), we obtain, for $\xi$ sufficiently large, that

$$L \geqslant L_2 = \tfrac{1}{9}\lambda^{\frac{3}{2}}\xi^{\frac{1}{2}} - 3/\lambda^{\frac{3}{2}} - \tfrac{1}{9}\eta\lambda^{\frac{3}{2}}. \tag{5.19}$$

Thus $L \to \infty$ as $\xi \to \infty$.

*Part* (*b*) The unbounded oscillations do not satisfy (5.18). In this case, we subdivide the problem into ranges of $\xi$ for which the bound (5.18) is alternately satisfied and not satisfied. Thus, for $\xi$ sufficiently large, we have

$$L \geqslant \min\{L_1, L_2\}.$$

Here, $L_1$ is defined by (5.17) and represents the behaviour of $L$ in those intervals of $\xi$ (sufficiently large) in which (5.18) is not satisfied. $L_2$ is defined by (5.19) and represents the behaviour of $L$ in those intervals of $\xi$ in which (5.18) is satisfied. Since both $L_1$ and $L_2 \to \infty$ as $\xi \to \infty$ the desired result follows. □

## 6. Existence theorem

In this section, we prove the central existence theorem for nontrivial solutions of the free-boundary problem.

THEOREM 6.1 *For $0 < \lambda < \lambda_c$ and $f(U) = U^2$, there exist an odd number, greater than or equal to one, of solutions of the free-boundary problem defined by equations* (2.1)–(2.3).

*Proof.* We proved in Section 3 that the free-boundary problem is equivalent to a shooting problem defined by the zeros of the parameter-dependent functional (3.1). Further, we proved in Corollary (3.2) that $G(\xi)$ is well defined for each value of $\xi \geqslant \eta$. Now, $G(\eta) = 0$ and, by Lemma 5.1,

$$\lim_{\xi \to \eta_+} G_\xi(\xi) = -\infty$$

for $\lambda < \lambda_c$. Thus, if we can establish that

$$\lim_{\xi \to \infty} G(\xi) > 0, \tag{6.1}$$

then, by continuity, we will have demonstrated the existence of an odd number ($\geqslant 1$) of zeros of $G(\xi)$ and hence solutions of the free-boundary problem. Thus, we now establish that (6.1) holds for $\lambda > 0$.

We define $P(x)$ by

$$P(x) = \exp\left(\int_0^x Q(\tau)\, d\tau\right).$$

By using equation (2.1), this may be written as

$$P(x) = \exp\left(\int_0^x \int_0^\tau \lambda f(U(t))\, dt\, d\tau\right). \tag{6.2}$$

Using this definition of $P(x)$, equation (2.2) becomes

$$(PU')' + Pf(U) = 0.$$

Integrating once gives us

$$U'(x) = -\int_0^x \frac{P(y)}{P(x)} f(U(y))\, dy.$$

Thus, formally integrating (2.1), we obtain

$$\lambda_c U'(x) + Q(x) = \int_0^x \left(\lambda - \lambda_c \frac{P(y)}{P(x)}\right) f(U(y))\, dy.$$

Setting $z = y/x$ yields

$$\lambda_c U'(x) + Q(x) = x \int_0^1 \left(\lambda - \lambda_c \frac{P(xz)}{P(x)}\right) f(U(xz))\, dz. \tag{6.3}$$

Since, for $\lambda > 0$ and $f(U) = U^2$, Lemma 5.5 demonstrates that $L \to \infty$ as $\xi \to \infty$, we may choose $\xi > \xi^*$ such that

$$L(\xi) > \max\left\{2\left(\frac{2\lambda_c}{f(\eta)}\right)^{\frac{1}{2}} \frac{1}{\lambda}, \left(\frac{2 \ln(\lambda_c/\lambda)}{\lambda f(\eta)}\right)^{\frac{1}{2}}\right\}. \tag{6.4}$$

For $\xi > \xi^*$, we prove that the function

$$R(z) = \lambda - \lambda_c \frac{P(Lz)}{P(L)}$$

changes sign once and only once for $0 \leq z \leq 1$. From (6.2) we have

$$\frac{P(Lz)}{P(L)} = \exp\left(-\int_{Lz}^L \int_0^\tau \lambda f(U(t))\, dt\, d\tau\right).$$

Thus, since $f(U(t))$ is strictly positive for $0 \leq t \leq L$, we deduce that $R(z)$ is a monotonically decreasing function of $z$. Also

$$\frac{P(0)}{P(L)} = \exp\left(-\int_0^L \int_0^\tau \lambda f(U(t))\, dt\, d\tau\right) \leq \exp\left(-\int_0^L \int_0^\tau \lambda f(\eta)\, dt\, d\tau\right),$$

since $f(\xi) \geq f(U(t)) \geq f(\eta)$ for $0 < t < L$. Thus

$$P(0)/P(L) \leq \exp[-\tfrac{1}{2}\lambda f(\eta) L^2].$$

By (6.4), we obtain

$$R(0) = \lambda - \lambda_c P(0)/P(L) \geq \lambda - \lambda_c \exp[-\tfrac{1}{2}\lambda f(\eta) L^2] > 0$$

for $\xi$ sufficiently large. Also, $R(1) = \lambda - \lambda_c < 0$. Thus $R(z)$ changes sign once and only once for $0 \leq z \leq 1$. We define $z^*$ by $R(z^*) = 0$ and denote $U(Lz^*)$ by $U^*$. By (6.3), we have

$$\lambda_c U'(L) + Q(L) = L \int_0^1 R(z) f(U(Lz))\, dz.$$

Since $f(U)$ $(= U^2)$ is a monotonically increasing function of its argument and since, by Lemma 3.1, $U(Lz)$ is monotonically decreasing for $0 \leq z \leq 1$, we obtain the inequality

$$\lambda_c U'(L) + Q(L) \geq L \int_0^1 R(z) f(U^*)\, dz. \tag{6.5}$$

Now

$$R(z) = \lambda - \lambda_c \exp\left(-\int_{Lz}^{L}\int_0^{\tau} \lambda f(U(t))\,dt\,d\tau\right).$$

Since $f(U(t)) \geq f(\eta)$ for $0 \leq t \leq L$, we obtain, for $0 \leq z \leq 1$,

$$R(z) \geq \lambda - \lambda_c \exp\left[\tfrac{1}{2}\lambda f(\eta)L^2(z^2-1)\right] \geq \lambda - \lambda_c \exp\left[\tfrac{1}{2}\lambda f(\eta)L^2(z-1)\right].$$

Thus,

$$\int_0^1 R(z)\,dz \geq \int_0^1 \lambda - \lambda_c \exp\left(\frac{\lambda f(\eta)}{2}L^2(z-1)\right) dz$$

$$= \lambda - \lambda_c \frac{2}{\lambda f(\eta)L^2}\left[\exp\left(\frac{\lambda f(\eta)}{2}L^2(z-1)\right)\right]_0^1 > \lambda - \frac{2\lambda_c}{\lambda f(\eta)L^2}.$$

By (6.4) we deduce that, for $\xi$ sufficiently large,

$$\int_0^1 R(z)\,dz > \tfrac{3}{4}\lambda.$$

Substituting this into (6.5), we obtain

$$G(\xi) = \lambda_c U'(L) + Q(L) \geq \tfrac{3}{4}\lambda L f(U^*) \geq \tfrac{3}{4}\lambda L f(\eta),$$

for $\xi$ sufficiently large. Thus, for $\lambda > 0$ and $f(U) = U^2$, Lemma 5.5 gives

$$\lim_{\xi \to \infty} G(\xi) = \infty.$$

Hence (6.1) holds and the theorem is complete.

## 7. Conclusions

In conclusion, we have proved the nonexistence of nontrivial solutions of the free-boundary problem defined by equations (2.1)–(2.3) for all $\lambda \notin (0, \lambda_c)$. In the case $f(U) = U^2$ (which arises in practice), we have proved the existence of a *global* branch of nontrivial solutions for all $\lambda \in (0, \lambda_c)$. As with many global existence results, the method of proof is *nonconstructive*; the result is important, however, since it validates the numerical predictions contained in [2]. A *local constructive* approach to the existence theory for $\lambda \sim \lambda_c$ is developed in [2].

We note here that numerical evidence [2] strongly suggests that the nontrivial solutions found for $\lambda \in (0, \lambda_c)$ are unique. The proof of this conjecture would, however, be a nontrivial matter.

The extension of the existence proof to more general nonlinearities $f(U)$ relies on proving Lemma 5.5 for different functions $f(U)$; it has been pointed out to the author by Dr. J. Norbury that a dynamical-systems approach to this lemma might be fruitful. However, since it is the case $f(U) = U^2$ which is of practical importance, this possibility has not been explored further.

From the scalings described in Section 1 and the two Theorems 4.1 and 6.1, we deduce the following results about the behaviour and form of combustion waves for small driving velocities. The combustion will be typified by large zones of chemical reaction (since $L \sim \mu^{-\frac{1}{4}} L_0$) which propagate slowly (since $Q \sim \mu^{\frac{1}{4}} Q_0$). The temperature variations within the combustion zone will be of $O(1)$ with respect to the inlet gas velocity. Further, since the parameter $\lambda$ is linearly related to the specific heat of the combustible solid, the nonexistence Theorem 4.1 indicates that there are upper and lower limits on the range of solid reactant specific heats above and below which combustion cannot be sustained. Since this nonexistence theorem requires only that $f(U)$ be a positive function, we deduce that the upper and lower bounds are independent of the relationship between the reaction rate (1.1) and the gas temperature $W$. This relationship can only be determined experimentally, and so the result is of some importance.

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