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THE GLOBAL ATTRACTOR UNDER DISCRETISATION

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ABSTRACT. The effect of temporal discretisation on dissipative differential equations is analysed. We discuss the effect of discretisation on the global attractor and survey some recent results in the area. The advantage of concentrating on ω and α limit sets (which are contained in the global attractor) is described. An analysis of spurious bifurcations in the ω and α limit sets is presented for linear multistep methods, using the time-step Δt as the bifurcation parameter. The results arising from application of local bifurcation theory are shown to hold globally and a necessary and sufficient condition is derived for the non-existence of a particular class of spurious solutions, *for all* $\Delta t > 0$. The class of linear multistep methods satisfying this condition is fairly restricted since the underlying theory is very general and takes no account of any inherent structure in the underlying differential equations. Hence a method complementary to the bifurcation analysis is described, the aim being to construct methods for which spurious solutions do not exist *for* Δt *sufficiently small*; for infinite dimensional dynamical systems the method relies on examining steady boundary value problems (which govern the existence of spurious solutions) in the singular limit correpsonding to $\Delta t \to 0_+$. The analysis we describe is helpful in the design of schemes for long-time simulations.

1 Introduction

The real world presents a variety of dynamical phenomena of bewildering complexity. These phenomena can often be modelled by means of differential equations. However, differential equations can rarely be solved in closed form and so it is often necessary to replace them by finite dimensional maps. These maps can exhibit a wide range of dynamical behaviour but it is not always clear which, if any, of the numerical observations are related to the real world. A necessary step in ascertaining the relationship of numerically generated dynamics to the real world is to study the dynamical properties of differential equations and their discretisations in conjunction. This is the approach taken here. The solutions of nonlinear maps are usually far easier to generate than the solutions of nonlinear differential equations and the price we pay is that the behaviour of the maps is often far more complicated than that of the underlying differential equations. Thus it is important to design schemes in which the effect of numerical artefacts on the dynamics is minimised.

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It is well known that a numerical method for an initial value problem which is convergent at a fixed time does not necessarily yield the same asymptotic behaviour as the underlying initial value problem, for fixed values of the time-step. For linear ordinary differential equations whose solutions decay with time it is necessary to operate the numerical method in the region of absolute stability of the scheme in order to obtain the correct asymptotic behaviour. This condition is stronger than that of zero stability, which is required for convergence. The issue of absolute stability and the construction of A-stable methods has been central to the development of numerical methods for linear initial value problems.

It is only relatively recently that analogous problems have been studied for the discretisation of nonlinear evolution equations, particularly those involving partial differential operators. Early work of interest includes the paper of Burrage and Butcher [2] in which AN and BN stability are defined for contractive nonlinear ODEs. The essential difficulty with the nonlinear problem is the dependence on initial conditions. The current growth of interest in the dynamics of numerical methods for nonlinear problems has been fuelled by the input of many ideas from dynamical systems. Here we discuss some of these recent results in a unified dynamical systems framework. The advantage of a dynamical systems approach to the numerical analysis of initial value problems is that it forces consideration of the flow generated by the numerical method. This is in contrast to classical numerical analysis which focusses on convergence of individual trajectories from a fixed initial condition.

In this paper we concentrate on dissipative differential equations: for simplicity we will take this to mean problems for which all trajectories converge into an absorbing set in the phase space after a finite time. (This definition is suitable in finite dimensions but sometimes inadequate in infinite dimensions [17].) In section 2 we review some definitions from continuous dynamical systems. Section 3 contains analogous definitions for discrete dynamical systems and a discussion of the relevance of these concepts to numerical analysis. It is shown, by means of an example, that the destruction of a global attractor under discretisation can occur when the unstable manifold of a spurious solution introduced by discretisation is connected to infinity. In section 4 we analyse spurious bifurcations in the ω and α limit sets for linear multistep methods. The limit sets are contiained in the global attractor, if it exists. The analysis yields conditions necessary and sufficient for the non-existence of a particular class of spurious solutions, for all values of the discretisation parameter Δt. A complementary analysis, which applies only for Δt sufficiently small, is described and applied to examples from discretisation of reaction-diffusion equations.

2 Continuous Evolution Semigroups

Consider an ODE in a Banach space

$$u_t = G(u), \tag{1}$$

together with an initial condition on u at $t = 0$. Here $G(u) : B \to B'$ for two Banach spaces B and B' where, typically, $B \subset B'$. We now define the evolution semigroup $S(t)$ which maps the solution at a given time to a solution t units of time later. This and subsequent definitions in sections 2 and 3 can be found in Temam [17].

Definition 2.1 *A strongly continuous evolution semigroup is a mapping* $S(t) : B \to B$ *for some Banach space* B *with* $S(t)$ *satisfying*

$$S(t+s) = S(t) \circ S(s) \ \forall t, s \geq 0, \tag{2}$$

$$S(0) = I, \tag{3}$$

and

$$S(t)u \ \text{continuous in} \ t \in [0, \infty) \ \text{for each} \ u \in B. \tag{4}$$

The solution $u(t)$ *of (1) then satisfies*

$$u(t) = S(t)u(0), \tag{5}$$

where $u(0)$ *is the initial condition. The action of* $S(t)$ *on a set* $D \subseteq B$ *is defined by*

$$S(t)D = \bigcup_{u(0) \in D} S(t)u(0).$$

The possible asymptotic states of $u(t)$ are captured in the ω and α limit sets which can be defined in terms of the evolution semigroup $S(t)$ as follows. Note that the α limit set of a point is not defined in general since infinite dimensional dissipative dynamical systems are usually not defined backwards in time – consider the heat equation for example. However, the α limit set may exist for specific choices of $u(0)$.

Definition 2.2 *The* ω *limit set and the* α *limit set (when it exists) of a point* $u(0)$ *are defined respectively by*

$$\omega(u(0)) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)u(0)} \tag{6}$$

and

$$\alpha(u(0)) = \bigcap_{s \leq 0} \overline{\bigcup_{t \leq s} S(-t)^{-1}u(0)}. \tag{7}$$

The ω *and* α *limit sets of a set* $D \subseteq B$ *are defined analogously with* D *replacing* $u(0)$.

Discussion Consider equation (6): we take the union of segments of trajectories starting at time $t \geq s$; then, taking the intersection over all $s \geq 0$, we eliminate the transient behaviour and we are left with information about the asymptotics of the evolution semigroup. Typical members of the ω and α limit sets include *steady solutions, periodic solutions, quasi-periodic solutions* and *strange attractors.*

We can now define the global attractor. This, if it exists, is a compact attractor which attracts the bounded sets of B uniformly and whose basin of attraction is the whole space B. The global attractor is essentially comprised of members of the ω and α limit sets of points, together with the trajectories which connect them. Detailed discussion of the global attractor can be found in [7,9,17].

Definition 2.3 *An attractor for the semigroup* $S(t)$ *is a set* $A \subset B$ *satisfying the following properties:*

(i) A is a postively and negatively invariant set for the semigroup. (A postively and negatively invariant set satisfies $S(t)A = A, \forall t \geq 0$.)
(ii) A possesses an open neighbourhood U such that $dist(S(t)u(0), A) \to 0$ as $t \to \infty$, for all $u(0) \in U$. The distance from a point to a set is found by taking the infimum over the distances to all points in the set.
The largest open set satisfying (ii) is known as the basin of attraction of A.

Definition 2.4 *The global attractor is a compact attractor A which satisfies*

$$d(S(t)U, A) \to 0 \text{ as } t \to \infty$$

uniformly for any bounded set $U \subset B$. Here

$$d(A_1, A_2) = \sup_{x \in A_1} \inf_{y \in A_2} d(x, y).$$

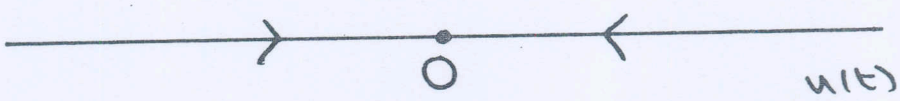
The basin of attraction for A is the whole of B.

Examples 1 We study a very simple example which will become interesting under discretisation. Consider the ODE

$$u_t = -u^3, \tag{8}$$

with $u(0) \in \Re$. The global attractor for this differential equation is the singleton $\{0\}$ for which the conditions of Definition 2.4 are easily checked. The dynamics of (8) can be summarised as follows:

Figure 1. The dynamics of (8)

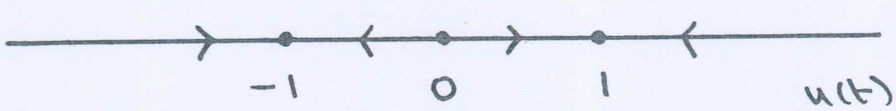


A slightly more interesting example is the ODE

$$u_t = u - u^3, \tag{9}$$

with $u(0) \in \Re$. Here the global attractor consists of the three equilibria 0, 1 and -1 together with the heteroclinic orbits connecting 0 to -1 and to 1. The dynamics of (9) can be summarised as follows:

Figure 2. The dynamics of (9)



The first step towards proving the existence of a global attractor is the contruction of an *absorbing set* which all trajectories starting in a bounded set enter in a finite time. For a precise definition see [17]. The existence of an absorbing set is a *necessary* condition for the existence of a global attractor. Theorem 1.1 in Chapter I of [17] describes additional conditions required to ensure that the existence of an absorbing set is *sufficient* for the existence of a global attractor. Clearly it is important to determine what happens to an absorbing set under discretisation. We show in the next section that spurious members of the ω and α limit sets can destroy the absorbing set property. Consequently we study the existence of spurious solutions in section 5.

3 Discrete Evolutions Semigroups

Consider now a nonlinear map in a Banach space

$$U_{n+1} = \Phi(U_n), \tag{10}$$

together with an initial condition on U_0. Here $\Phi : B \to B$ for a Banach space B. We are particularly interested in the case where (10) forms an approximation to (1); we refrain from making a specific identification between U_n and $u(t)$ because this will depend on the nature of the discretisation (whether or not it is a one step method, whether or not the elements of an infinite dimensional Banach space are approximated finite dimensionally *etc.*) Definitions analogous to 2.1,2.2,2.3 and 2.4 can be made as in [17]. Througout this section n and m denote integers.

Definition 3.1 *A discrete evolution semigroup is a mapping $S_n : B \to B$ for some Banach space B with S_n satisfying*

$$S_{n+m} = S_n \circ S_m \;\forall\; integer\; n, m \geq 0 \tag{11}$$

and

$$S_0 = I. \tag{12}$$

The solution U_n of (10) then satisfies

$$U_n = S_n U_0. \tag{13}$$

The action of S_n on a set $D \subseteq B$ is defined by

$$S_n D = \bigcup_{U_0 \in D} S_n U_0. \tag{14}$$

Definition 3.2 *The ω limit set and the α limit set (when it exists) of a point U_0 are defined repsectively by*

$$\omega(U_0) = \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} S_n U_0} \tag{15}$$

and

$$\alpha(U_0) = \bigcap_{m \leq 0} \overline{\bigcup_{n \leq m} S_{-n}^{-1} U_0} \tag{16}$$

The ω and α limit sets of a set $D \subseteq B$ are defined analogously with D replacing U_0.

Definition 3.3 *An attractor for the semigroup S_n is a set $A \subset B$ satisfying the following properties:*
(i) A is a positively and negatively invariant set for the semigroup. (A positively and negatively invariant set satisfies $S_n A = A$ $\forall n \geq 0$.)
(ii) A possesses an open neighbourhood U such that $dist(S_n U_0, A) \to 0$ as $t \to \infty$, for all $U_0 \in U$. The distance from a point to a set is found by taking the infimum over the distances to all points in the set.
The largest open set satisfying (ii) is known as the basin of attraction of A.

Definition 3.4 *The global attractor is a compact attractor A which satisfies*

$$d(S(t)U, A) \to 0 \text{ as } t \to \infty$$

uniformly for any bounded set $U \subset B$. Here

$$d(A_1, A_2) = \sup_{x \in A_1} \inf_{y \in A_2} d(x, y).$$

The basin of attraction for A is the whole of B.

Let us assume now that the discrete evolution semigroup S_n forms an approximation to the continuous evolution semigroup $S(t)$ and that Δt is the discretisation parameter. In the case where the underlying problem is infinite dimensional we shall suppress explicit reference to spatial discretisation, assume that a mesh refinement path has been chosen and assume that a suitable prolongation operator has been chosen. There are three fundamental questions which confront the numerical analyst:

(i) Are the ω and α limit sets for $S(t)$ and S_n the same, or "close", in particular as $\Delta t \to 0$?

(ii) If $S(t)$ has a (global) attractor A does S_n have a (global) attractor $A_{\Delta t}$?

(iii) If the answer to (ii) is "yes", then does $A_{\Delta t} \to A$ as $\Delta t \to 0$?

The first question has been studied by a number of workers: in [6] conditions are derived which ensure that no spurious steady solutions are introduced by Runge-Kutta discretisation. In [10] the existence of spurious steady solutions is examined for Runge-Kutta, linear mulitstep and predictor-corrector methods. Spurious periodic solutions are often introduced by discretisation; in particular, spurious period 2 solutions in n for the discrete semigroup are important for discretisations of evolution equations whose linear variational equations have real eigenvalues. Examples of this are given in [16], the background is surveyed in [14] and a complete theory described in [15,16]. Spurious invariant curves are also important and an instructive example of this is given in [1]. A unified approach to the existence of spurious members of the ω and α limit sets, using bifurcation theory, is contained in [11]. Some recent work by Elliott [5] describes a class of time-discretisation methods which preserve the Liapunov functional structure of certain evolution equations; this powerful approach prevents the existence of almost all spurious solutions, except spurious steady solutions introduced by *spatial* discretisation.

The second and third questions are more difficult and less work has been done on them. Kloeden and Lorenz examine the approximation of attracting sets in ODEs by one-step time-discretisations [12] and by multistep methods [13]. Hale and co-workers [8] have examined similar problems by different means. For a survey of further results see [9, p170.]

The answer to the first question is fundamental to the second and third questions since the ω and α limit sets of points are necessarily contained in the global attractor. Often a global attractor is destroyed by discretisation because there are orbits connecting a spurious member of the limit sets to infinity and hence the absorbing set property no longer holds; an example of this is given below.

In this paper we concentrate on question (i). Question (i) is related to both (ii) and (iii) and sheds light on those problems (see the example below). Furthermore, question (i) makes sense in those problems for which the dynamical system does not possess a global attractor, such as Hamiltonian systems and PDEs whose solutions blow-up in finite time.

Example 2 Consider the ODE (8) under discretisation by the Euler method. We obtain

$$U_{n+1} = U_n - \Delta t U_n^3. \tag{17}$$

Note that the steady solution of the differential equation, 0, is preserved under discretisation. However a period 2 solution is introduced:

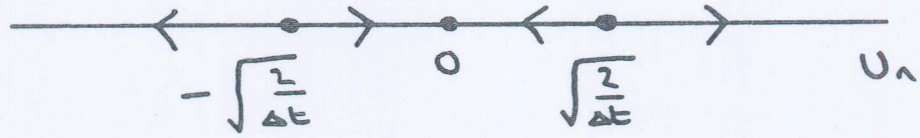
$$U_n = \sqrt{\frac{2}{\Delta t}}(-1)^n \tag{18}$$

is a solution of (17).

The spurious solution plays a very important role since it divides the phase space into regions in which the correct asymptotic behaviour is observed ($U_n \to 0$ as $n \to \infty$) and in which the scheme blows-up and solutions diverge to infinity. This is simple to see. Let $U_c = \sqrt{\frac{2}{\Delta t}}$. If $|U_0| > U_c$ then $1 - \Delta t U_0^2 < -1$. Hence, by (17) we have $|U_1| > |U_0|$. By induction, noting that there are no fixed points $> U_c$ of the map: $|U_n| \to |U_{n+1}|$, we deduce that $U_n \to \infty$ as $n \to \infty$. Similarly, if $|U_0| < U_c$ then $1 > 1 - \Delta t U_0^2 > -1$ and so $|U_1| < |U_0|$. By induction we deduce that $U_n \to 0$ as $n \to \infty$, since 0 is the only fixed point $< U_c$ of the map: $|U_n| \to |U_{n+1}|$.

We have constructed orbits connecting the spurious solution to 0 and to ∞. Thus the absorbing set property, which is necessary for the existence of a global attractor, has been destroyed by the unstable spurious solution. This spurious solution exists for any finite $\Delta t > 0$. Hence a global attractor does not exist for the discretisation. The dynamics of the map (17) are summarised in the following Figure which should be compared with Figure 1 for the underlying ODE:

Figure 3. The dynamics of (17)



It is worth pointing out that, although the period 2 solution is obviously spurious and is also unstable it is still fundamental to an understanding of the equation (8) under the discretisation (17). Note also that the spurious solution approaches infinity in norm as $\Delta t \to 0$; this is typical of spurious solutions and is proved under quite general conditions in [16]. □

Further simple examples of spurious solutions can be found in section 2 of [16]. In higher dimensions the role of unstable spurious solutions is very similar to that demonstrated in the example above. A precise characterisation of the destination of all initial data is generally impossible but it is often the case that unstable spurious solutions have unstable manifolds which connect with infinity, thereby destroying the absorbing set property. See [16, Theorem 5.2] for an example of this in high dimensions. Hence, for the numerical solution of evolution equations where the long-time dynamics are of interest and where large classes of initial conditions are considered, it is very important to design schemes which minimise the effect of spurious members of the limit sets.

There are two approaches to this design criterion which we will consider here. Both rely conceptually on the idea of treating the approximating dynamical system as a bifurcation problem, with the discretisation parameter Δt playing the role of bifurcation parameter. The first approach is to design schemes for which there are no spurious members (of a particular type) of the limit sets for *all* $\Delta t > 0$. This correponds to proving a global non-existence result for a particular class of branches of solutions. Unfortunately, this sometimes leads to schemes which are impractical for other reasons – solution of the nonlinear algebraic equations is prohibitively expensive, a maximum principle is difficult to enforce *etc.* A second approach is to establish that branches of a particular class of spurious solutions cannot extend back to *arbitrarily small positive* Δt. We consider both these approaches in the following section, where linear multistep methods are examined in detail.

4 Spurious Bifurcations In The ω and α Limit Sets

In this section we consider linear multistep methods for the solution of (1). These can be written in the general form

$$\sum_{k=0}^{M} \alpha_k U_{n+k} = \Delta t \sum_{k=0}^{M} \beta_k G(U_{n+k}). \tag{19}$$

Here U_k approximates $u(k\Delta t)$. (Note that, to formulate this method as a one-step map in the form (10), it is necessary establish the solvability of the nonlinear equation for U_{n+M}

(if $\beta_M \neq 0$) and to consider a vector containing M steps as a single unknown; see [11].) In this paper we consider only the effect of time-discretisation, although (1) may, of course, be a system of ODEs arising from the spatial discretisation of a PDE. We assume that (19) forms a consistent approximation of (1). The following polynomials will be useful.

Definition 4.1 *We define*

$$\rho(z) = \sum_{k=0}^{M} \alpha_k z^k \text{ and } \sigma(z) = \sum_{k=0}^{M} \beta_k z^k. \tag{20}$$

Definition 4.2 below is a generalisation of a definition contained in [6]. Roughly, *regular of degree 1* means no spurious steady solutions in the limit sets and *regular of degree 2* means no period 2 solutions in the limit sets. Note that period 2 solutions are always spurious and their importance in determining the dynamics of discretisations has been illustrated by means of example. See [16] for further examples. One could consider spurious solutions of higher periodicity but period one (steady) and period two solutions are particularly important since they bifurcate from steady solutions of the map as Δt varies and are observed generically in systems with hyperbolic equilibria which are of saddle or of nodal type. Spurious invariant curves are also of interest since they are observed in systems with hyperbolic equilibria of spiral type – see [11].

Definition 4.2

The numerical method (19) is regular of degree 1 if every fixed point $\hat{U} \in B$ of the map (19) satisfies $G(\hat{U}) = 0$, for all $\Delta t > 0$ and for all equations (1).

The numerical method (19) is regular of degree 2 if (19) does not admit period 2 solutions in n for all $\Delta t > 0$ and all equations (1).

Theorem 4.3 characterises the regularity of linear multistep methods.

Theorem 4.3

(i) *The numerical method (19) is regular of degree 1.*

(ii) *If $\rho(-1) \neq 0$ the numerical method (19) is regular of degree 2 if and only if $\sigma(-1) = 0$.*

(iii) *If $\rho(-1) = 0$ the numerical method (19) is not regular of degree 2.*

Proof Part (i) is proved in [10]. Parts (ii) and (iii) are proved in [11]. For motivation we shall sketch the proof of (ii) for the θ method only; this simplifies the technicalities considerably without losing the central ideas of the proof. The θ method is

$$U_{n+1} - U_n = \Delta t[(1-\theta)G(U_n) + \theta G(U_{n+1})]. \tag{21}$$

Here $\rho(z) = z - 1$ and $\sigma(z) = (1-\theta) + \theta z$. Note that $\rho(-1) \neq 0$ and that $\sigma(-1) = 1 - 2\theta$. Let $U_{2n} = U$ and $U_{2n+1} = V$. Then period 2 solutions are pairs U, V both in B with $U \neq V$ satisfying

$$V - U = \Delta t[(1-\theta)G(U) + \theta G(V)] \tag{22}$$

and

$$U - V = \Delta t[(1-\theta)G(V) + \theta G(U)]. \tag{23}$$

Note that the steady solutions of the differential equation (1) are solutions of (22,23) with $U = V$.

To prove the *only if* part of (ii) we show that period 2 solutions bifurcate from steady solutions if $\sigma(-1) \neq 0$. Assume that (1) has a steady solution $\hat{U}$. Without loss of generality we let $\hat{U} = 0$ so that $G(0) = 0$. Note that $U = V = 0$ satisfies (22,23). Let $dG(0)$, the Frechet derivative of G at 0, be non-singular and have a real, non-zero, simple eigenvalue η. Then a little calculation shows that the Frechet derivative of the system (22,23) is singular at

$$\Delta t = \frac{2}{(2\theta - 1)\eta} \; \textit{and} \; U = V = 0. \tag{24}$$

Futhermore, this eigenvalue Δt is simple; thus we deduce from Theorem 5.3 in Chapter 5 of [4] that period 2 solutions of (19) bifurcate from the trivial solution for $\theta \neq \frac{1}{2}$ (*i.e.* $\sigma(-1) \neq 0$.)

To prove the *if* part of (ii) subtract (23) from (22). This gives $V = U$ if $\theta = \frac{1}{2}$ (*i.e.* $\sigma(-1) = 0$) and so period 2 solutions do not exist. □

The result of Theorem 4.3 would seem to suggest that the optimal choice of linear multistep method is one for which $\sigma(-1) = 0$; in particular, for the θ method this requires $\theta = \frac{1}{2}$. (We restrict ourselves to discussion of the θ method henceforth.) However, often considerations other than spurious members of the limit sets come into play. For parabolic problems the maximum principle is of great importance and the choice $\theta = 1$ has many advantages from this point of view. For semilinear parabolic equations, the convergence of local attractors is proved in [8; Theorem 4.1] under the condition $\theta \in (\frac{1}{2}, 1]$ and the extra dissipativity afforded by this choice underlies the proof.

Thus the choice $\theta = \frac{1}{2}$ is often *not* made and it is important to examine what happens to branches of spurious solutions in this case. In general we know that branches of spurious solutions will exist if $\theta \neq \frac{1}{2}$ since they bifurcate from genuiune equilibria at the the critical value of Δt given by (24). Thus it is important to design schemes for which the branches of spurious solutions cannot extend back to Δt arbitrarily small. We indicate how the analysis of steady boundary value problems can shed light on the design of schemes appropriate to a particular equation. This is illustrated by means of two examples from the discretisation of reaction-diffusion equations.

Example 3 Consider the equation

$$u_t = u_{xx} - u^p, \tag{25}$$

where p is odd. For simplicity consider the Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0. \tag{26}$$

The solution of this problem is defined for all time and the global attractor is the trivial solution 0. Let us assume that we apply the θ method to (25). Thus period 2 solutions satisfy (22,23) where $G(u) = u_{xx} - u^p$. Since p is odd we can examine the existence of period 2 solutions satisfying the Z_2 symmetry $V = -U$; equation (22) yields

$$\Delta t(1-2\theta)[U_{xx} - U^p] + 2U = 0, \tag{27}$$

with boundary conditions

$$U(0) = U(1) = 0. \tag{28}$$

We are interested in choosing schemes for which there are no solutions of (27,28) as $\Delta t \to 0_+$. Multiplying by U and integrating by parts we obtain

$$-\Delta t(1-2\theta)[\int_0^1 U_x^2 + U^{p+1}dx] + 2\int_0^1 U^2 dx = 0. \tag{29}$$

If $\theta > \frac{1}{2}$ the left-hand side is positive for Δt positive and hence no spurious solutions exist for $\Delta t > 0$. On the other hand, if $\theta < \frac{1}{2}$ it may be shown that solutions of (27,28) can exist for Δt arbitrarily small and positive [3]. Hence $\theta > \frac{1}{2}$ is a superior choice to $\theta < \frac{1}{2}$. Note that this result is in accordance with Theorem 4.1 of [8], alluded to above; it shows that the convergence of attractors is intimately related to the non-existence of spurious solutions. □

In practice equation (25) will be discretised in space as well as in time. However, the semi-discrete argument given sheds light on the appropriate choice of fully discrete scheme: we now show that any solution sequence satisfying the backward Euler scheme ($\theta = 1$) coupled with the usual centred differences in space must converge to zero as $n \to \infty$. *That is, the global attractor is preserved under discretisation.* This is strongly related to the fact that the backward Euler scheme does not possses period 2 solutions in n.

Example 3 Continued Let u_j^n denote our approximation to $u(j\Delta x, n\Delta t)$. Set $J\Delta x = 1$ and $r = \frac{\Delta t}{\Delta x^2}$. The backward Euler scheme coupled with centred differences in space yields, for $j = 1, \ldots, J-1$

$$u_j^{n+1} = u_j^n + r\delta^2 u_j^{n+1} - \Delta t(u_j^{n+1})^p. \tag{30}$$

The boundary conditions are

$$u_0^n = u_J^n = 0. \tag{31}$$

Let

$$u_{max}^n = \max_{0\le j\le J} u_j^n \tag{32}$$

and

$$u_{min}^n = \min_{0\le j\le J} u_j^n. \tag{33}$$

Re-arranging equation (30) gives u, for $1 \le j \le J-1$,

$$\begin{aligned} u_j^{n+1}[1 + 2r + \Delta t(u_j^{n+1})^{p-1}] &\le u_j^n + ru_{j+1}^{n+1} + ru_{j-1}^{n+1} && (34)\\ &\le u_{max}^n + 2ru_{max}^{n+1}. && (35) \end{aligned}$$

If u^{n+1}_{max} is attained for $1 \le j \le J-1$ we find that

$$[1 + \Delta t(u^{n+1}_{max})^{p-1}]u^{n+1}_{max} \le u^n_{max}. \tag{36}$$

If u^{n+1}_{max} is attained for $j = 0, J$ then clearly $u^{n+1}_{max} = 0$. Similarly, by considering $-u^n_j$ we can show that, if u^{n+1}_{min} is attained for $1 \le j \le J-1$,

$$[1 + \Delta t(u^{n+1}_{min})^{p-1}]u^{n+1}_{min} \ge u^n_{min}. \tag{37}$$

If u^{n+1}_{min} is attained for $j = 0, J$ then clearly $u^{n+1}_{min} = 0$. Combining (36,37) and the fact that the maximum (resp. minimum) is non-negative (resp. non-positive) we deduce that $u^n_j \to 0$ as $n \to \infty$. (We use the fact that p is odd.) Thus the global attractor has been preserved under discretisation by the backward Euler scheme. Such a result *cannot be proved for the forward Euler scheme* ($\theta = 0$) without introducing a restriction on Δt in terms of the initial data – see Theorem 5.2 in [16]. Such a restriction is necessary to avoid the effect of the spurious period 2 solutions shown to exist for $\theta = 0$. □

Example 3 is particularly simple since we could show non-existence of spurious solutions for *all* $\Delta t > 0$ when $\theta > \frac{1}{2}$. In general this is not possible since for $\theta > \frac{1}{2}$ spurious periodic solutions always bifurcate from linearly unstable equilibria at positive values of Δt. (This follows from equation (24) with $\eta > 0$ which implies that the equilibrium is unstable; note that in Example 3 the only equilibrium is 0 and it is stable so that bifurcation does not occur). However, it is always be possible to write down a boundary value problem governing the existence of spurious period 2 solutions. Analysing the existence of solutions for this problem in the limit $\Delta t \to 0_+$ can yield practical guidelines for the choice of scheme. We illustrate this by means of a more involved example.

Example 4 Consider the equation

$$u_t = u_{xx} - f(u), \tag{38}$$

where

$$f(u) = \sum_{j=0}^{p} a_j u^{2j+1}. \tag{39}$$

The boundary conditions are

$$u(0,t) = u(1,t) = 0. \tag{40}$$

Here $a_p > 0$; under this assumption, it is shown in Chapter III of [17] that the problem (38-40) possesses a global attractor. We now show that the backward Euler method is an appropriate discretisation of this equation since branches of spurious period 2 solutions cannot extend back to Δt arbitrarily small. The argument is easily extended to cope with all $\theta \in (\frac{1}{2}, 1]$.

The backward Euler discretisation of (38) gives

$$U^{n+1} - U^n = \Delta t[U^{n+1}_{xx} - f(U^{n+1})], \tag{41}$$

with boundary conditions

$$U^n(0) = U^n(1) = 0. \tag{42}$$

Seeking period 2 solutions with the Z_2 symmetry $U^{2n+1} = -U^{2n} = U$ we obtain

$$-U_{xx} + f(U) + \frac{2}{\Delta t}U = 0, \tag{43}$$

with boundary conditions

$$U(0) = U(1) = 0. \tag{44}$$

It is our aim to show that (43,44) has no non-trivial solutions for Δt sufficiently small and positive.

Multiplying (43) by U and integrating by parts we obtain

$$\int_0^1 [U_x^2 + Uf(U) + \frac{2}{\Delta t}U^2]dx = 0. \tag{45}$$

The Young inequality gives us

$$\frac{1}{2}a_p U^{2p+2} - c \le Uf(U).$$

Using this in (45) we obtain

$$\int_0^1 U^2 dx \le \frac{c\Delta t}{2} \text{ and } \int_0^1 U^{2p+2}dx \le \frac{2c}{a_p}. \tag{46}$$

Applying the Hölder inequality we obtain a further bound

$$\int_0^1 U^2 dx \le (2c/a_p)^{\frac{1}{p+1}}. \tag{47}$$

Whilst (46) and (47) are useful in establishing where spurious solutions can be found in function and parameter space, they are not sufficient to establish non-existence for Δt sufficiently small. This we now do.

Equation (45) together with (39) gives

$$0 \ge \int_0^1 U^2[\frac{2}{\Delta t} + \sum_{j=0}^{p} a_j U^{2j}]dx \tag{48}$$

Because $a_p > 0$ it follows that the sum is bounded from below independently of Δt, since the $a_j's$ do not involve Δt. Hence, by choosing Δt sufficiently small, the integrand can be made positive and we deduce that non-trivial solutions of (43,44) cannot exist.

5 Conclusions

We have examined dissipative evolution problems and their discretisations in a unified fashion. It has been shown that the property of possessing a global attractor can be destroyed by the introduction of spurious memebers of the ω and α limit sets when discretisation is performed. This destruction of the global attractor occurs when the unstable manifold of the spurious solution is connected to infinity. The existence of spurious solutions in linear multistep methods has been examined by two methods both of which treat the numerical method as a dynamical system parameterised by the time-step Δt. The first corresponds to a global theory on the existence of branches of spurious solutions for *all values of* Δt. It is based around the examination of bifurcation of spurious solutions from genuiune members of the limit sets. The second method corresponds to an analysis of the existence of spurious solutions in the singular limit $\Delta t \to 0_+$: it relies on examining boundary value problems and proving the non-existence of solutions *for* Δt *sufficiently small.*

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