2

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Volterra integral equations and a new Gronwall inequality (Part II: The nonlinear case)

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Synopsis

We consider nonlinear singular Volterra integral equations of the second kind. We generalise the transformation method introduced in Part I of this paper [**6**] to cope with both the nonlinearity and slightly more general singular kernels. We also consider a particular class of nonlinear equation for which the solution behaviour is known. Using this *a priori* knowledge, we propose a modification of the transformation technique which results in a numerical method with good asymptotic stability properties. Applying the general theory of Part I of this paper, we prove convergence of this scheme.

1. Introduction

In this paper we are concerned with the application of numerical methods to the solution of weakly singular Volterra equations of the form

$$y(t)=g(t)+\int_0^t \frac{K(t,s,y(s))}{(t-s)^\alpha}\,ds, \tag{1.1}$$

where $0<\alpha<1$. The application of the usual polynomial spline collocation leads to poorly convergent numerical schemes [**2**], and so we introduce a variable mesh to overcome this difficulty [**5**], [**6**].

In Section 2 we define a wide class of nonlinear equations of the form (1.1) for which the solution exists for all t. Using this knowledge, we tailor our transformations, which eliminate the singularity and introduce the variable mesh, to allow for the behaviour of the solution at infinity. Under certain assumptions on the behaviour of the numerical approximation, which are borne out in practice, we prove convergence of the scheme in Section 3. In Section 4 we discuss the large time behaviour of our numerical approximation.

In Section 5 we return to the general equation (1.1) and discuss the appropriate transformations necessary to reduce the equation to a regular form. We believe that the method discussed has a fairly wide range of application in the study of numerical methods for weakly singular integral equations. It relies on being able to spot the appropriate transformations which reduce weakly singular equations into a regular form. Using the error bounds of Part I of this paper, convergence of the numerical schemes based on this transformation approach can then be proved.

2. A particular class of nonlinear equations

We are interested in finding $y(t)$, for $t \geqq 0$, where $y(t)$ satisfies

$$y(t) = 1 - \int_0^t (t-s)^{-\frac{1}{2}} K(y(s))\, ds \quad \text{for} \quad t \geqq 0. \tag{2.1}$$

We assume that $K(y)$ satisfies $K(\beta) = 0$ and $K_y(y) > 0$ and also that $K(y) \in C^2[\beta, 1]$. In the Appendix we prove a number of results regarding the behaviour of the solution to equation (2.1), including its existence and uniqueness. We now describe these results.

Initially both $y(t)$ and $K(y(t))$ decrease from their starting values of 1 and $K(1)$, respectively. At the origin the derivative $y'(t) = O(t^{-\frac{1}{2}})$ as $t \to 0$ and so the initial decrease is fairly rapid. Thereafter the solution behaviour is regular and $y(t)$ tends to a limiting value of β as $t \to \infty$. For finite values of t, $y(t)$ may be shown to satisfy $\beta < y(t) \leqq 1$ and computations indicate that, in fact, $y(t)$ decreases monotonically from $y(0) = 1$ to $y(\infty) = \beta$.

The class of equation (1.1) was first studied in the particular case

$$K(y) = 1 - (\beta/y)^{\nu} \tag{2.2}$$

which arises from an industrial problem involving coupled parabolic partial differential equations **[7]**. It was in this context that Norbury **[5]** introduced the transformation technique subsequently generalised in both parts of this paper.

As in Part I of this paper **[6]**, we introduce a transformation

$$s = t \sin^2 \phi \tag{2.3}$$

under which equation (2.1) becomes

$$y(t) = 1 - \int_0^{\pi/2} 2\sqrt{t} \sin \phi K(t \sin^2 \phi))\, d\phi. \tag{2.4}$$

For the linear equation in **[6]** we introduced a transformation $t = \theta^2$ at this stage to remove the singularity of the solution at the origin. This and transformation (2.3) are appropriate for all non-linear equations of the form (1.1) with $\alpha = \frac{1}{2}$ for which the solution may not exist for all time. However, for the particular case defined by equation (2.1), the solution is known to exist for all time and so we may make the transformation

$$t = \tan^2 \theta, \tag{2.5}$$

which maps the infinite range of t onto a finite range of θ. On defining $u(\theta) \equiv y(t)$, equation (2.4) becomes

$$u(\theta) = 1 - \int_0^{\pi/2} 2 \tan \theta \,.\, \sin \phi K(u(\phi^*))\, d\phi \tag{2.6}$$

for $0 \leqq \theta \leqq \pi/2$, where

$$\tan \phi^* = \tan \theta \sin \phi. \tag{2.7}$$

As for the linear case detailed in **[6]**, these two transformations have regularised the behaviour of the solution (for $0 \leqq \bar{\theta} < \pi/2$) and hence we are in a position to apply standard product integration techniques to solve equation (2.6). Defining $h = \pi/2(N+1)$, we divide the interval $0 \leqq \theta \leqq \pi/2$ into $N+1$ equal subintervals by the $N+2$ equally-spaced points $\theta_i = ih$ for $i = 0, \ldots, N+1$. We next divide the interval $0 \leqq \phi \leqq \pi/2$ into i unequal subintervals defined by the $(i+1)$ unequally spaced points ϕ_{ij} for $j = 0, \ldots, i$ defined by

$$\sin \phi_{ij} = \tan \phi_j / \tan \theta_i. \tag{2.8}$$

Hence equation (2.6) may be written as

$$1 - u(\theta_i) = \sum_{j=1}^{i} \int_{\phi_{ij-1}}^{\phi_{ij}} 2 \tan \theta_i \sin \phi K(U(\phi^*))\, d\phi. \tag{2.9}$$

By virtue of the non-uniform subdivision of the range of integration ϕ, the

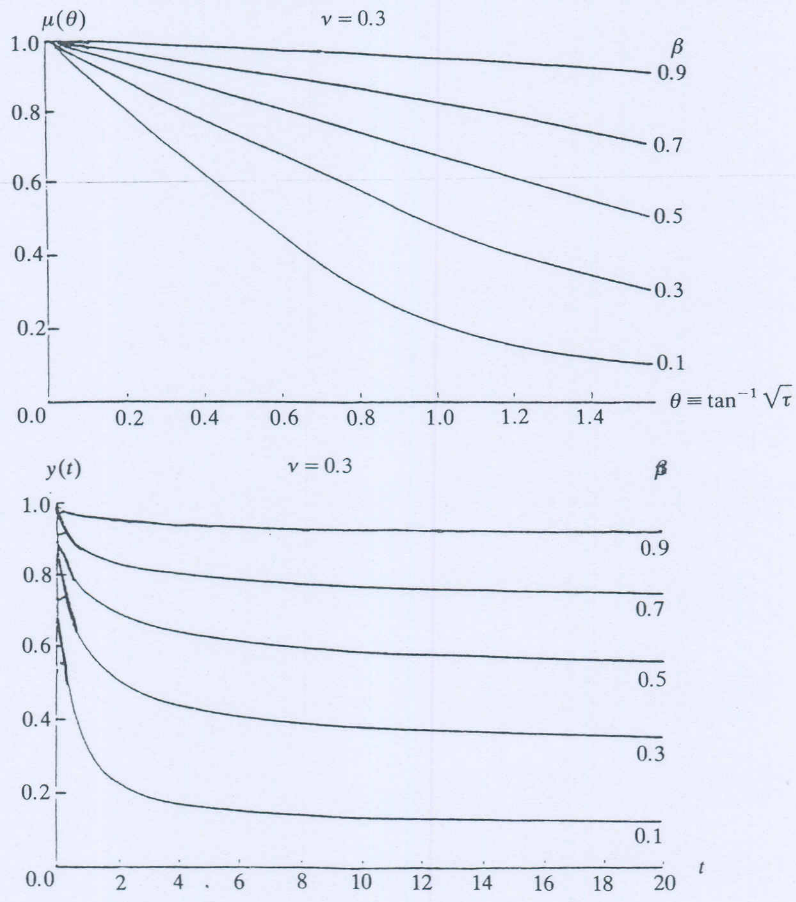

Figure 1.

unknown in the integrand, $u(\phi^*)$, assumes values $u(\theta_j)$ at the end of each range of integration. We now introduce a trapezium approximation to the kernel K of equation (2.9): defining u_j to be our numerical approximation to the true solution $u(\theta_j)$, we obtain

$$1 - u_i = \sum_{j=1}^{i} \int_{\phi_{ij-1}}^{\phi_{ij}} 2 \tan \theta_i \sin \phi \frac{[(\phi_{ij} - \phi)K(u_{j-1}) + (\phi - \phi_{ij-1})K(u_j)]}{(\phi_{ij} - \phi_{ij-1})}. \quad (2.10)$$

Finally, we can perform the integrations exactly to obtain

$$1 - u_i = 2 \tan \theta_i \sum_{j=1}^{i} a_{ij}K(u_{j-1}) + b_{ij}K(u_j), \quad (2.11)$$

where a_{ij} and b_{ij} are defined in terms of ϕ_{ij} by equations (3.9) and (3.10) in **[6]**. The resulting non-linear algebraic equation (2.11) is solved for u_i by Newton iteration, using u_{i-1} as the initial guess. In practice it is found that one Newton iteration is sufficient to achieve the accuracy of the scheme. Figure 1 shows typical solution profiles, for the case defined by equation (2.2), in both the natural independent variable t and the transformed variable θ.

3. Convergence

THEOREM 3.1. *Suppose that the function K in equation* (2.1) *satisfies $K(\beta) = 0$ and $K_y(y) > 0$ and also that $K \in C^2[\beta, 1]$. Suppose further that the exact solution u_i of the nonlinear algebraic equation* (2.11) *satisfies $\beta \leqq u_i \leqq 1$ for $i = 1, \ldots, N$. Then the exact solution u_i to the discrete equation* (2.11) *is convergent to the solution $u(\theta_i)$ $(\theta_i \leqq \bar{\theta} < \pi/2)$ of the continuous equation* (2.6) *with order $h^{\frac{3}{2}-\varepsilon}$, where ε is an arbitrarily small positive number.*

Note on Theorem. The convergence result requires that u_i should satisfy $\beta \leqq u_i \leqq 1$ for each $i = 1, \ldots, N$. As mentioned in Section 2, the true solution does satisfy $\beta \leqq u(\theta_i) \leqq 1$ and in practice we have found that the solution scheme proposed in Section 2 does generate solutions u_i which satisfy $\beta \leqq u_i \leqq 1$.

The proof of Theorem 3.1 requires the following four results which we now prove (Lemmas 3.1–3.4). Since the d_{ij}'s defined by (3.4) are proportional to the d_{ij}'s defined by (3.21) of **[6]** and since the definition of the ϕ_{ij}'s in this paper is a functional generalisation of those defined in **[6]** (compare (2.8) of this paper with (3.5) of **[6]**) the proofs of these lemmas are analagous to those in **[6]** and hence the details are omitted.

Throughout the following four lemmas d_{ij} is as defined in (3.4) and its iterates $d_{ij}^{(\mu)}$ and the function $f^{(\mu)}(s, t)$ are as defined by **[6**, (2.7–2.9)**]**.

LEMMA 3.1. *The variable $h \sum_{j=1}^{i-1} d_{ij}^{(1)}$ is bounded above independently of h.*

Proof. Since $1 - hd_{ii} = 1$ and since $2 \tan \theta_i$ is independent of h, it is sufficient to show that

$$S = \sum_{j=1}^{i-1} (\phi_{ij+1} - \phi_{ij-1})$$

is bounded independently of h. In a manner analogous to that employed in **[6**, Lemma 3.1], we can show that

$$S \leqq \int_{\theta_1}^{\theta_{i-1}} \frac{2 \sec^2 (\theta + h)\, d\theta}{[\tan^2 \theta_i - \tan^2 (\theta + h)]^{\frac{1}{2}}} \leqq \pi.$$

Hence, noting that hd_{ii-1} is $O(h^{\frac{1}{2}})$, the result follows.

LEMMA 3.2. *The second iterated kernel $d_{ij}^{(2)}$ generates a function $f^{(2)}((j+1)h, ih)$ which satisfies the conditions that $f^{(2)}(t, t)$ exists, $\dfrac{\partial f^{(2)}}{\partial s}(s, t) \leqq 0$ and $\dfrac{\partial f^{(s)}}{\partial s}(s, t) \geqq 0$.*

Proof. As in [6, Lemma 3.2] we see that it suffices to consider the sum

$$S = \sum_{\ell=j+1}^{i-1} (\phi_{i\ell+1} - \phi_{i\ell-1})(\phi_{\ell j+1} - \phi_{\ell j-1}),$$

for an upper bound on S will define $f^{(2)}(s, t)$ up to a multiplicative constant. By use of the results in [6, Lemma 3.2] we can show that

$$S \leqq \frac{4h \sec^2 \theta_i}{\tan \theta_{i-1}} \mathbb{K}(1 - 1/c^2),$$

where $c = \tan \theta_{i-1}/\tan \theta_{j+1}$.

Again following [6, Lemma 3.2], we define

$$f^{(2)}(s, t) = M\mathbb{K}(1 - \tan^2 s/\tan^2 t), \tag{3.1}$$

where $\mathbb{K}$ is a complete elliptic integral of the second kind (see [1] p. 596, p. 590) and where M is independent of h. By using the documented properties of $\mathbb{K}$ (see the graph in [1] p. 592) namely that $\mathbb{K}(0) = \pi/2$ and $\mathbb{K}'(x) \geqq 0$, we see that the conditions of the lemma are satisfied.

LEMMA 3.3. *The function $f^{(2)}(s, t)$ defined in (3.1) is integrable with respect to s, over the range $0 \leqq s \leqq t$, and the result is bounded independently of h.*

Throughout this lemma $\mathbb{K}$ is as defined in the proof of Lemma 3.2.

Proof. Define $J = \int_0^t f^{(2)}(s, t)\, dt$.

By substituting $\tan s = v \tan t$ and noting that $\sec^2 s \geqq 1$, we obtain

$$J \leqq M \tan t \int_0^1 \mathbb{K}(1 - v^2)\, dv.$$

We use the results of Lemma 3.3 in [6], and obtain

$$J \leqq \frac{M \tan t}{\sqrt{2}} \mathbb{K}^2(1/\sqrt{2}).$$

Thus we have the desired result.

LEMMA 3.4. *The consistency error, ce_i, satisfies*

$$ce_i \leqq \delta = M_3 h^{\frac{5}{2}-\varepsilon}.$$

Throughout this lemma, M_k, $K=1, 2, 3$ represent numbers bounded above independently of h.

Proof. Similarly to [6, Lemma 3.4], we can show that

$$ce_i = \sum_{j=1}^{i} \int_{\phi_{ij-1}}^{\phi_{ij}} \tan\theta_i \frac{d^2}{d\phi^2}(K(u(\phi^*)))_{\phi^*=\eta}(\phi-\phi_{ij-1})(\phi-\phi_{ij}) \sin\phi \, d\phi.$$

Because of the smoothness of the solution $u(\theta)$ to equation (2.6) [4] and because $K \in C^2[\beta, 1]$, we obtain

$$ce_i \leqq M_1 \sum_{j=1}^{i} (\phi_{ij} - \phi_{ij-1})^2(\cos\phi_{ij-1} - \cos\phi_{ij})$$

as in [6, Lemma 3.4]. Redefining $x = \tan\theta/\tan\theta_i$ and $y = \tan(\theta - h)/\tan\theta_i$ the analysis proceeds as in [6, Lemma 3.4]. Noting that $(x-y) \leqq M_2 h/\tan\theta_i$, we obtain the final result

$$ce_i \leqq M_3 h^{\frac{3}{2}-\varepsilon}.$$

Proof of Theorem 3.1. Defining $\varepsilon_i = (u_i - u(\theta_i))$ to be the approximation error at each mesh point, subtraction of equation (2.10) from (2.9) and application of the Mean Value Theorem gives us

$$\varepsilon_i \leqq \delta - \sum_{j=1}^{i} 2\tan\theta_i(a_{ij}K_y(\xi_{j-1})\varepsilon_{j-1} + b_{ij}K_y(\xi_j)\varepsilon_j), \tag{3.2}$$

where δ is an upper bound on the consistency error, defined in Lemma 3.4 and where ξ_j lies between u_j and $u(\theta_j)$. Since u_j and $u(\theta_j)$ are both greater than β, so is ξ_j. Hence $K_y(\xi_j)$ is bounded above independently of h and so we may re-write (3.2) as

$$|\varepsilon_i| \leqq (1 + b_{ij}K_y(\xi_j))\,|\varepsilon_i| \leqq \delta + \sum_{j=0}^{i-1} e_{ij}\,|\varepsilon_j|, \tag{3.3}$$

where

$$e_{i0} = 2\,M\tan\theta_i a_{i0}$$

and

$$e_{ij} = 2M\tan\theta_i(a_{ij+1} + b_{ij}), \quad j = 1, \ldots, i-1,$$

and M is positive and independent of h. As in [6], we can show, by use of the Mean Value Theorem, that

$$e_{ij} \leqq 2M\tan\theta_i(\phi_{ij+1} - \phi_{ij-1}). \tag{3.4}$$

Hence, since we may assume that $\varepsilon_0 = 0$, we have

$$|\varepsilon_i| \leqq \delta + h\sum_{j=1}^{i} d_{ij}\,|\varepsilon_j|,$$

where $hd_{ij} = 2M\tan\theta_i(\phi_{ij+1} - \phi_{ij-1})$, for $j = 1, \ldots, i-1$ and $hd_{ii} = 0$. This is in the form of [6, inequality (2.5)] and so we attempt to apply [6, Theorem 2.2]

Since $hd_{ii}=0$, condition (i) of this theorem is trivially satisfied. Lemmas 3.1, 3.2 and 3.3 show us, respectively, that conditions (ii), (iii) and (iv) of the theorem are satisfied. Hence we deduce from (3.3) that

$$\varepsilon_i \leqq C_1\delta,$$

where C_1 is independent of h. By Lemma 3.4 we obtain the final result

$$\varepsilon_i \leqq C_2 h^{\frac{3}{2}-\varepsilon} \tag{3.5}$$

where C_2 is also independent of h.

4. The large time behaviour of the numerical method

For equations of the form (2.1) we have described a transformation of independent variables which both eliminates the singularity present in the solution of equation (2.1) and which maps the infinite range of the independent variables onto a finite range. We then propose a solution technique for the transformed equations which corresponds to application of the product trapezium rule followed by use of Newton iteration to solve the resulting nonlinear algebraic equations. In Section 3 we have proved convergence of this method and Table 1 presents figures which demonstrate this.

TABLE 1
Numerical values for $u(0.706858=\tan^{-1}\sqrt{t})$ indicating convergence as $N\to\infty$.

N (β)	0.1	0.5	0.9
80	0.371314	0.770255	0.962498
160	0.372020	0.770476	0.962532
320	0.372290	0.770558	0.962545
640	0.372390	0.770588	0.962550

However, further to this property of convergence, we notice that in practice the method performs well in approximating the solution to equation (2.1) for arbitrarily large t. More precisely, we notice that as we repeatedly halve $h(=\pi/2(N+1))$, the division of the range $0\leqq\theta\leqq\pi/2$, our approximation u_N to the true solution $u(hn)$ remains accurate; not only does it continue to satisfy $u_N\geqq\beta$ but it also appears that $u_N\to\beta$ as $N\to\infty$ (with $h\to 0$ and $Nh\to\pi/2$). Since $u(\pi/2)=\beta$ (see Appendix) this property indicates that the proposed solution technique remains stable for arbitrarily large values of t (that is for θ arbitrarily close to $\pi/2$).

In other words it appears that in transforming and numerically approximating equation (2.1) in the manner described in Section 2, we have preserved the strongly attractive properties of the limiting solution β.

5. The general case ($0<\alpha<1$)

We now return to the general equation (1.1) and describe the appropriate transformations to reduce it to a regular form. These transformations are detailed in [**4**] and form the basis of the numerical method first introduced by Norbury [**5**]. As in Section 2, we transform s by defining

$$s = \tan^2 \phi \tag{5.1}$$

under which equation (1.1) becomes

$$y(t) = g(t) + \int_0^{\pi/2} K(t, t \sin^2 \theta, y(t \sin^2 \phi)) \,.\, 2t^{1-\alpha} \cos^{(1-2\alpha)} \phi \,.\, \sin \phi \, d\phi. \tag{5.2}$$

Assuming that α is a rational number, we write $\alpha = r/q$ where r and q are co-prime. (In the unlikely event that an irrational α should occur some form of rational approximation is necessary.) We now define θ, $u(\theta)$ and $h(\theta)$ by

$$\theta = t^{1/q} \tag{5.3}$$

$$u(\theta) \equiv y(t)$$

and

$$h(\theta) \equiv g(t).$$

Using transformation (5.3), equation (5.2) becomes

$$u(\theta) = h(\theta) + \int_0^{\pi/2} 2\theta^{q-r} \bar{K}(\theta^q, \phi^*, u(\phi^*)) \cos^{(1-2\alpha)} \phi \sin \phi \, d\phi, \tag{5.4}$$

where

$$\phi^* = \theta \sin^{2/q} \phi \tag{5.5}$$

and $\bar{K}(., \theta, .) \equiv K(., t, .)$. Whereas equation (1.1) possesses a derivative $y'(t) = 0(t^{-\alpha})$ as $t \to 0$, it may be shown [**4**] that the transformed equation (5.4) has a solution $u(\theta) \in C^m$, where $g(t) \in C^m$, if we assume sufficient differentiability of K. Hence we may apply standard product integration techniques to the solution of equation (5.4). We note that since such approximations will involve some form of interpolation of the kernel K, it is necessary that the integral I, defined by

$$I = \int_0^{\pi} \cos^{(1-2\alpha)} \phi \sin \phi \, d\phi, \tag{5.6}$$

exists in order that the resulting discrete equations are defined. This may be shown to be so by using the substitution $v = \cos \phi$, yielding

$$I = \frac{1}{2(1-\alpha)}$$

and since $0 < \alpha < 1$ the result is defined.

6. Conclusions

We have presented an approach to the numerical solution of weakly singular integral equations which relies on being able to spot the appropriate transformations which eliminate the singular behaviour of the solution. We have described these transformations for the most general weakly singular Volterra integral equations of the second kind (equation (1.1)), and for a particular class of equation (2.1) we have proved convergence of a numerical method based on these transformations. We have also demonstrated that our particular choice of discretisation results in desirable stability properties.

However, the rate of convergence of schemes based on these transformations has not been determined for general α, nor for higher order product integration rules than the trapezium rule which we consider in Section 3. As in [**6**] we believe that the rate of convergence in these more general cases is best analysed by using model linear problems.

Appendix

Consider the equation

$$y(t) = 1 - \int_0^t (t-s)^{-\frac{1}{2}} K(y(s))\, ds \tag{A1}$$

subject to the conditions

$$K(\beta) = 0, \tag{A2}$$

$$K_y(y) > 0, \tag{A3}$$

and

$$K(y) \in C^2[\beta, 1]. \tag{A4}$$

The solution $y(t)$ of equation (A1) subject to conditions (A2), (A3) and (A4) satisfies the following four propositions.

PROPOSITION 1. *The solution $y(t)$ of equation (A1) satisfies $\beta \leqq y(t) \leqq 1$.*

Proof. We prove that $y(t) \geqq \beta$ for all t. It then follows automatically from (A1) that $y(t) \leqq 1$. Suppose, for the sake of contradiction, that $y(t_1) = \beta$ and that $y(t) < \beta$ for $0 < t_1 < t \leqq t_2$. Then (A1) gives

$$1 - y(t_1) = \int_0^{t_1} (t_1 - s)^{-\frac{1}{2}} K(y(s))\, ds$$

and

$$1 - y(t_2) = \int_0^{t_2} (t_2 - s)^{-\frac{1}{2}} K(y(s))\, ds.$$

Subtracting these, we have

$$\begin{aligned} y(t_1) - y(t_2) &= \int_0^{t_2} (t_2 - s)^{-\frac{1}{2}} K(y(s))\, ds - \int_0^{t_1} (t_1 - s)^{-\frac{1}{2}} K(y(s))\, ds \\ &= \int_0^{t_1} [(t_2 - s)^{-\frac{1}{2}} - (t_1 - s)^{-\frac{1}{2}}] K(y(s))\, ds \\ &\quad + \int_{t_1}^{t_2} (t_2 - s)^{-\frac{1}{2}} K(y(s))\, ds. \end{aligned}$$

Now, in $0 \leqq s \leqq t_1$ the first integrand is negative, since $t_1 < t_2$ and $K(y(s)) \geqq 0$, while in $t_1 \leqq s \leqq t_2$ the second integrand is negative, since $K(y(s)) \leqq 0$.

Thus we have shown that $y(t_1)-y(t_2)<0$. As $y(t_1)=\beta$ this implies that $y(t_2)>\beta$, which contradicts our initial assumption. Thus $y(t)\geqq\beta$ for all $t>0$.

PROPOSITION 2. *There exists a unique solution $y(t)$ to equation $(A1)$ in $0\leqq t<\infty$.*

Proof. The result follows from [**3**, Lemma 2]. The necessary *a priori* bounds are provided in Proposition 1.

PROPOSITION 3. *The derivative $y'(t)$ of $y(t)$, the solution of equation $(A1)$, satisfies $y'(t)=-K(1)t^{-\frac{1}{2}}+0(1)$ as $t\rightarrow 0$.*

Proof. The result follows from [**4**, Section 5]. Note that the expression for K_1 should read

$$K_1=2g(0,f(0))(\nu+1-p)\dots(1-p)/(2\nu+2-2p).$$

PROPOSITION 4. *The solution $y(t)$ of equation $(A1)$ satisfies $y(t)\rightarrow\beta$ as $t\rightarrow\infty$.*

Proof. The result follows from [**3**, Theorem 3], with slight adjustment. (The conditions of the theorem may be suitably relaxed using the *a priori* bound $\beta\leqq y(t)\leqq 1$ from Proposition 1.)

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