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The Mathematics of Porous Medium Combustion

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Abstract. Two partial differential equations arising from the theory of porous medium combustion are examined. While both equations possess a trivial steady solution, the form of the reaction rate, which is discontinuous as a function of the dependent variable, precludes bifurcation of non-trivial steady solutions from the branch of trivial solutions. A constructive approach to the existence theory for non-trivial global solution branches is developed. The method relies on finding an appropriate set of solution dependent transformations which render the problems in a form to which local bifurcation theory is directly applicable. Specifically, by taking a singular limit of the (solution dependent) transformation, an artificial trivial solution (or set of solutions) of the transformed problem is created. The (solution dependent) mapping is not invertible when evaluated at the trivial solution(s) of the transformed problem; however, for non-trivial solutions which exist arbitrarily close to the artificial trivial solution, the mapping is invertible. By applying local bifurcation theory to the transformed problem and mapping back to the original problem, a series expansion for the non-trivial solution branch is obtained.

1. Introduction.

In this paper we analyze two time-dependent partial differential equation problems arising from the theory of porous medium combustion [**3**]. The first problem, (P1), defined in section 2, represents a simplified one-dimensional model of porous medium combustion when the depletion of oxygen during the reaction is insignificant. The model comprises a pair of reaction-diffusion equations coupled to a hyperbolic equation. The second problem, (P2), defined in section 3, is a single reaction-diffusion equation; it may be derived from (P1) as the first term in an asymptotic expansion of the governing equations in terms of two small parameters and as such it represents the evolution of temperature in highly exothermic combustion problems before the depletion of reactants becomes significant.

The novel mathematical aspect of the two problems is that the reaction rates are discontinuous as a function of the independent variables. This fact is reflected in the appearance of Heaviside step functions in the reaction terms. We present a unified approach to the existence theory for steady solutions of problems (P1) and (P2).

For problem (P1) we pose the equations on the whole real line and seek steady travelling wave solutions whereas for (P2) we examine steady solutions on a bounded interval. Both problems possess a trivial constant solution corresponding to an ambient state of no chemical reaction. However, the form of the reaction term in both (P1) and (P2) precludes the possibility of bifurcating non-trivial solutions in the neighborhood of the trivial solutions. Thus it is important to develop a constructive approach to the existence problem for both (P1) and (P2) in order that a starting point may be found for numerical studies of the two problems.

In sections 2 and 3 we define the two problems (P1) and (P2), respectively. In section 4 we describe an approach to the existence theory for (P1) and in section 5 we analyze this approach in more detail in the context of problem (P2).

2. Problem (P1).

The simplified model of porous medium combustion described here is derived in [**3**]. The unknowns σ, u and w represent the solid heat capacity and the solid and gas tempreatures respectively. The concentration of combustible solid may be determined as a linear function of the solid heat capacity. The governing equations are

$$\frac{\partial \sigma}{\partial t} = -\lambda r, \tag{2.1}$$

$$\sigma \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + w - u + r \tag{2.2}$$

$$\text{and } \delta \frac{\partial w}{\partial t} + \mu \frac{\partial w}{\partial z} = u - w, \tag{2.3}$$

where

$$r = \mu^{1/2} H(u - u_c) H(\sigma - \tau) f(w). \tag{2.4}$$

Here $H(X)$ is the Heaviside unit step function defined by

$$H(X) = \begin{cases} 1 & X > 0 \\ 0 & X \leq 0. \end{cases}$$

$f(w)$ is a strictly-positive $C^2(u_a, \infty)$ function where u_a is defined by (2.5). We take $z \in \mathbb{R}$ and impose the boundary conditions that

$$\lim_{z \to \infty} u(\pm z, t) = \lim_{z \to \infty} w(-z, t) = u_a, \tag{2.5}$$

where $u_a < u_c$.

We assume that the discontinuity induced by the reaction rate r in equation (2.2) is taken up entirely by the second spatial derivative of u so that $u(z,t)$ is a piecewise C^2 function of z and a C^1 function of time. Equation (2.1) implies that $\sigma(z,t)$ is a piecewise C^1 function of both z and t.

Taken together with initial conditions, equations (2.1–2.5) define the time-dependent problem (P1). We make no attempt to prove global existence of solutions for all time; we observe, however, that sharp rises in either of the temperatures u and w are naturally limited by the function σ which is non-increasing in time. Further aspects of the time-dependent behavior of problem (P1) are discussed in Norbury and Stuart [**6**].

3. Problem (P2).

Consider the partial differential equations (2.1–2.3). The reaction rate (2.4) may be rewritten as

$$r = \mu^{1/2} d H(u-u_c) H(\sigma-\tau) \bar{f}(w) \tag{3.1}$$

where we have extracted a factor d from the function $f(w)$ and redefined the reaction rate accordingly. We now analyze the partial differential equations defined by (2.1–2.3) and (3.1) in the distinguished limit $\mu \to 0$, $\lambda/\mu^{1/2} \to$ constant, $\delta/\mu^{1/2} \to$ constant, and $d\mu^{1/2} \to 1$. If we expand the solution in the form

$$\begin{aligned} \sigma &= \sigma_0 + O(\mu^{1/2}), \\ u &= u_0 + O(\mu^{1/2}) \\ \text{and } w &= w_0 + O(\mu^{1/2}) \end{aligned} \tag{3.2}$$

then, to first order in $\mu^{1/2}$ equations (2.1) and (2.3) give us

$$\frac{\partial \sigma_0}{\partial t} = 0 \text{ and } w_0 = u_0.$$

Assuming that initially $\sigma_0(z,0) =$ const $> \tau$ equation (2.2) gives, to first order

$$\sigma_0 \frac{\partial u_0}{\partial t} = \frac{\partial^2 u_0}{\partial z^2} + H(u_0 - u_c)\bar{f}(u_0). \tag{3.3}$$

We consider this equation posed on a finite domain and the appropriate boundary conditions are

$$u_0(\pm s, t) = u_a. \tag{3.4}$$

Rescaling the variables by setting

$$v = \frac{u_0 - u_a}{u_c - u_a}, \quad t = s^2 \sigma_0 \tau \text{ and } y = z/s,$$

equations (3.3) and (3.4) may be reformulated, for some appropriately defined $g(v)$, as the Dirichlet problem

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y^2} + H(v-1)g(v), \tag{3.5}$$

with boundary conditions

$$v(\pm 1, t) = 0. \tag{3.6}$$

Together with initial conditions, equations (3.5) and (3.6) form problem (P2). As for problem (P1) we make no attempt to prove global existence of solutions of (P2). Indeed for certain classes of function $g(v)$ (which are likely to arise in practice) finite or infinite time blow-up can occur. These cases are of particular interest since they determine the temporal and spatial scales on which the expansion procedure defined by (3.2) becomes invalid. These and other time-dependent questions are addressed in Norbury and Stuart [**4**].

4. Existence Theory for (P1).

We seek travelling wave solutions of the time-dependent problem defined by equations (2.1–2.5). We will assume that $\delta = 0$. However, this is merely done to clarify the exposition; the Theorems in section 4 are easily modified to cope with the case $\delta > 0$. Defining

$$x = z - ct, \quad Q(x) \equiv c\sigma(z,t), \quad U(x) \equiv u(z,t) \text{ and } W(x) \equiv w(z,t)$$

we obtain

$$\frac{dQ}{dx} = \lambda r \tag{4.1}$$

$$\frac{d^2U}{dx^2} + Q\frac{dU}{dx} + W - U + r = 0 \tag{4.2}$$

$$\text{and } \mu\frac{dW}{dx} = U - W, \tag{4.3}$$

where

$$r = \mu^{1/2}H(U - u_c)H(Q - c\tau)f(W). \tag{4.4}$$

Here $0 < \tau < 1$. Conditions (2.5) transform to

$$\lim_{x\to\infty} U(\pm x) = \lim_{x\to\infty} W(-x) = u_a. \tag{4.5}$$

Assuming that no chemical reaction has taken place at $z = +\infty$ we deduce that σ will be determined by its initial concentration there, so that

$$\lim_{z\to\infty} \sigma(z,t) = 1.$$

Hence the appropriate boundary condition for Q is

$$\lim_{x\to\infty} Q(x) = c. \tag{4.6}$$

The system of equations (4.1–4.6) forms a nonlinear eigenvalue problem for the four-dimensional vector-valued function (Q, U, U', W) and the parameter c. Henceforth we denote this eigenvalue problem by (EVP). If we define the space PC^n to be the set of functions which are piecewise C^n on the whole real line, then we seek solutions (Q, U, W, c) of (EVP) which lie in the space $PC^1 \times PC^2 \times PC^1 \times \mathbb{R}$.

It is clear that (EVP) possesses the family of trivial solutions

$$Q \equiv c \text{ and } U \equiv W \equiv u_a, \text{ for any } c \in \mathbb{R}.$$

It may be demonstrated that all other solutions satisfy $\|U\|_\infty \geq u_c > u_a$. Consequently local bifurcation theory is not directly applicable to the problem since non-trivial solution branches do not exist arbitrarily close to the trivial solution branches. In this section we develop a constructive approach to the existence theory for nontrivial solution branches of (EVP). The first theorem, however, is a non-existence result.

THEOREM 4.1. *Solutions of (EVP) can exist only for* $0 < \lambda < (u_c - u_a)^{-1}$.

PROOF: Integrating equation (4.2) with respect to x over the whole real line and employing equation (4.3) we obtain

$$\int_{-\infty}^{+\infty} Q\frac{dU}{dx}\,dx + \int_{-\infty}^{+\infty} r\,dx = 0,$$

since bounded travelling wave solutions have $w(\infty) = u_a$. Integrating by parts and applying the boundary conditions (4.5) and (4.6) gives us

$$u_a\big[Q(+\infty) - Q(-\infty)\big] + \int_{-\infty}^{+\infty}\left(r - \frac{dQ}{dx}U\right)\,dx = 0.$$

However, combining this with equation (4.1) implies that

$$\int_{-\infty}^{+\infty}\big[1 + \lambda(u_a - U)\big]r = 0.$$

Since r is strictly positive on some non-zero interval of x, and since $u_c \leq U \leq \infty$ on this interval, we deduce that

$$0 < \lambda < (u_c - u_a)^{-1}. \tag{4.8}$$

This completes the proof.

□

We now aim to determine where solution branches exist within the allowable parameter regime defined by Theorem 4.1; in particular we shall attempt to determine the ends of solution branches. Since the reaction rate is zero outside a finite interval of x-space the equations defining (EVP) may be integrated explicitly in these regions. By so doing it is possible to convert (EVP) into a two-point free boundary problem posed on a finite domain. There are two distinct cases to consider dependent upon whether the reaction rate r, defined by (4.4), becomes zero at its left-hand end because U attains its critical value u_c, or because Q attains its critical value τc. We make the following definitions.

DEFINITION 4.2. *We define the reaction zone for EVP to be the region of x-space $(0, L)$ in which $r \neq 0$.*

DEFINITION 4.3. *We define a (U, U) switch solution to be a solution of EVP which satisfies $U(0) = U(L) = u_c$.*

DEFINITION 4.4. *We define a (Q, U) switch solution to be a solution of EVP which satisfies $Q(0) = \tau c$ and $U(L) = u_c$.*

There are two distinct free boundary problems governing the two distinct types of solutions defined above. The details of their derivation are described in Norbury and Stuart [**5**]. We now treat the two cases separately.

Free Boundary Problem 1; (U, U) solutions.

We denote the following free boundary problem by (FBP1). Find $(Q, U, W, \alpha, L) \in C^1(0, L) \times C^2(0, L) \times C^1(0, L) \times R \times R$, where

$$\begin{aligned} \frac{dQ}{dx} &= \lambda\mu^{1/2} f(W), \\ \frac{d^2U}{dx^2} &+ Q\frac{dU}{dx} + W - U + \mu^{1/2} f(W) = 0 \\ \text{and } \mu\frac{dW}{dx} &= U - W, \end{aligned}$$

together with the initial conditions

$$\begin{aligned} Q(0) &= \frac{\mu - \alpha - \mu\alpha^2}{1 + \mu\alpha}, \\ U(0) &= u_c, \\ \frac{dU}{dx}(0) &= (u_c - u_a)\alpha \\ \text{and } W(0) &= u_a + \frac{(u_c - u_a)}{1 + \mu\alpha}, \end{aligned}$$

and matching conditions

$$\begin{aligned} U(L) &= u_c \\ \text{and } \frac{dU}{dx}(L) &+ (u_c - u_a)Q(L) - \mu[W(L) - u_a] = 0. \end{aligned}$$

THEOREM 4.5. *Solutions of (FBP1) are in a one-to-one correspondence with solutions of (EVP) iff the solutions of (FBP1) satisfy $L > 0$ and*

$$\tau Q(L) \leq Q(0) < \mu < Q(L). \tag{4.9}$$

PROOF: See Norbury and Stuart [5].

□

The formulation of EVP as a free boundary problem on a finite domain has a decided advantage: by rescaling the independent variable we derive a problem to which local bifurcation theory is directly applicable. Under the transformation $s = x/L$ (FBP1) becomes

$$\frac{dQ}{ds} = \lambda \mu^{1/2} L f(w), \tag{4.10}$$

$$\frac{d^2U}{ds^2} + LQ\frac{dU}{ds} + L^2(W-U) + \mu^{1/2}L^2 f(W) = 0 \tag{4.11}$$

$$\text{and } \mu\frac{dW}{ds} = L(U-W). \tag{4.12}$$

The initial conditions become

$$Q(0) = \frac{\mu - \alpha - \mu\alpha^2}{1+\mu\alpha}, \tag{4.13}$$

$$U(0) = u_c \text{ and } \frac{dU}{ds}(0) = (u_c - u_a)\alpha L \tag{4.14}$$

$$\text{and } W(0) = u_a + \left(\frac{u_c - u_a}{1+\mu\alpha}\right). \tag{4.15}$$

The matching conditions are

$$U(1) = u_c \tag{4.16}$$

$$\text{and } \frac{dU}{ds}(1) + L(u_c - u_a)Q(1) - \mu L[W(1) - u_a] = 0. \tag{4.17}$$

We denote the problem defined by equations (4.10–4.17) by (FBP1*).

THEOREM 4.6. *Solutions of (FBP1*) which satisfy $L > 0$ are in a one-to-one correspondence with solutions of (FBP1).*

PROOF: If $L > 0$ then the mapping between the two problems is a bijection.

□

Examination of (FBP1*) shows that it possesses the family of trivial solutions

$$Q \equiv \frac{\mu - \bar{\alpha} - \mu\bar{\alpha}^2}{1+\mu\bar{\alpha}}, \quad U \equiv u_c, \quad W \equiv u_a + \frac{(u_c - u_a)}{1+\mu\bar{\alpha}}, \quad \alpha = \bar{\alpha}$$
$$\text{and } L = 0,$$

for all $\bar{\alpha} \in \mathbb{R}$. None of these trivial solutions correspond to a true travelling wave solutions since they do not satisfy $L > 0$ and (4.9). However, we may now apply local bifurcation theory to determine the possible location of bifurcation points from the family of trivial solutions. The most important of these trivial solutions is the one corresponding to $\bar{\alpha} = 0$, namely

$$Q \equiv \mu, \quad U \equiv W \equiv u_c, \quad \alpha = 0 \text{ and } L = 0, \tag{4.18}$$

since it is the only solution which, when perturbed by an arbitrarily small amount, can satisfy (4.9).

THEOREM 4.7. *Consider the trivial solution (4.18). Then a necessary condition for bifurcation from this solution into non-trivial solutions of (FBP1) is* $\lambda = \lambda_c = (u_c - u_a)^{-1}$.

PROOF: Let $\underline{\psi} = (\psi, \theta, \phi, \xi, \eta)$ represent the linearization of (Q, U, W, L, α) about the trivial solution (4.18). We consider the solution and its linearization as elements of the Banach space $C^1(0,1) \times C^2(0,1) \times C^1(0,1) \times \mathbb{R} \times \mathbb{R}$. Then $\underline{\psi}$ satisfies

$$\frac{d\psi}{ds} = \xi \lambda \mu^{1/2} f(u_c) \text{ and } \frac{d^2\theta}{ds^2} = \frac{d\phi}{ds} = 0$$

where the initial conditions are

$$\begin{aligned} \psi(0) &= -(1+\mu^2)\eta, \\ \theta(0) &= \frac{d\theta}{ds}(0) = 0 \\ \text{and } \phi(0) &= -(u_c - u_a)\mu\eta. \end{aligned}$$

The matching conditions are

$$\begin{aligned} &\theta(1) = 0 \\ &\text{and } \xi \int_0^1 [1 + \lambda(u_a - u_c)] \mu^{1/2} f(u_c)\, ds = 0. \end{aligned} \tag{4.19}$$

Here we have replaced (4.17) by the equivalent condition [5]

$$L \int_0^1 [1 + \lambda(u_a - u(s))] \mu^{1/2} f(w) ds$$

which may be derived by applying the method of proof employed in Theorem 4.1 to (FBP1); see [5].

This linear problem for $\underline{\psi}$ defines the Frechet derivative of (FBP1*) with respect to the trivial solution (4.18). In the case $\lambda \neq \lambda_c = (u_c - u_a)^{-1}$ the null-space of the Frechet derivative is one-dimensional and spanned by the eigenfunction

$$(4.20) \qquad (\psi, \theta, \phi, \xi, \eta) = \left(-(1+\mu^2), 0, -(u_c - u_a)\mu, 0, 1\right).$$

This eigenfunction corresponds to bifurcation into the trivial branch of solutions parametrized by $\bar{\alpha}$. However, in the case $\lambda = \lambda_c$, the null-space of the Frechet derivative is two-dimensional and spanned by (4.20) and a second eigenfunction

$$(\psi, \theta, \phi, \xi, \eta) = (\lambda_c \mu^{1/2} f(u_c)s, 0, 0, 1, 0).$$

This extra eigenfunction derives from the non-invertibility of the expression (4.19) for ξ at $\lambda = \lambda_c$. Consequently $\lambda = \lambda_c$ is the only point at which bifurcation into non-trivial solutions can occur.

□

That bifurcation actually does occur at $\lambda = \lambda_c$ may be shown by constructing a series expansion for the solution of (FBP1) in powers of $L \ll 1$. The details may be found in [**5**]. Thus the reformulation of (EVP) as a free boundary problem and the rescaling technique employed subsequently has led us to a constructive approach to the existence of non-trivial travelling wave solutions.

Free Boundary Problem 2; (Q, U) solutions.

We denote the following free boundary problem by (FBP2). Find $(Q, U, W, u_b, \alpha, L) \in C^1(0, L) \times C^2(0, L) \times C^1(0, L) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ where

$$(4.21) \qquad \frac{dQ}{dx} = \lambda \mu^{1/2} f(W)$$

$$(4.22) \qquad \frac{d^2U}{dx^2} + Q\frac{dU}{dx} + W - U + \mu^{1/2} f(W) = 0$$

$$(4.23) \qquad \text{and } \mu \frac{dW}{dx} = U - W,$$

together with initial conditions

$$Q(0) = \frac{\mu - \alpha - \mu\alpha^2}{1 + \mu\alpha}, \tag{4.24}$$

$$U(0) = u_b \text{ and } \frac{dU}{dx}(0) = (u_b - u_a)\alpha \tag{4.25}$$

$$\text{and } W(0) = u_a + \left(\frac{u_b - u_a}{1 + \mu\alpha}\right), \tag{4.26}$$

and matching conditions

$$Q(0) = \tau Q(L) \tag{4.27}$$

$$U(L) = u_c \tag{4.28}$$

$$\text{and } \frac{dU}{dx}(L) + (u_c - u_a)Q(L) - \mu[W(L) - u_a] = 0. \tag{4.29}$$

THEOREM 4.8. *Solutions of (FBP2) are in a one-to-one correspondence with solutions of (EVP) iff the solutions satisfy* $\alpha > 0$, $L > 0$, $Q(0) < \mu < Q(L)$ *and* $U(0) \geq u_c$.

PROOF: See Norbury and Stuart [5].

□

As for (FBP1) the reformulation of (EVP) as a free boundary problem has a number of analytical advantages. These are particularly clear in the case $\tau = 0$. In this case the boundary conditions (4.27) and (4.24) reduce to

$$Q(0) = 0 \tag{4.30}$$

$$\text{and } \alpha = \frac{-1 + [1 + 4\mu^2]^{1/2}}{2\mu}, \tag{4.31}$$

respectively.

Solutions of the simplified version of (FBP2) defined by (4.21–4.23), (4.25), (4.26), and (4.28–4.31) are characterized by a form of bifurcation from infinity as $\mu \to 0$. By this we mean there exists a branch of non-trivial solutions of (FBP2) at least one component of which approaches infinity as the critical value of the bifurcation parameter is approached. As a result the eigenvalue problem determining the existence of the bifurcating branch in the neighborhood of the bifurcation point is *nonlinear* in character. (A similar example of bifurcation from infinity in the case where the value of the bifurcation *parameter* at which bifurcation occurs is infinite may be found in [7]. In that case the leading order eigenvalue problem in the neighborhood of the bifurcation point is also nonlinear.)

For μ small a dimensional analysis shows that

$$Q \sim \mu^{1/4} Q_0, \quad U \sim U_0, \quad W \sim W_0, \quad u_b \sim U_{b0} \text{ and } L \sim \mu^{-1/4} L_0$$

where a subscript zero indicates an order one quantity. If we seek a series expansion of the reduced version of (FBP2) with $\tau = 0$ in powers of $\mu^{1/4} \ll 1$, then the leading order problem for $(Q_0, U_0, W_0, U_{b0}, L_0) \in C^1(0, L_0) \times C^2(0, L_0) \times C^1(0, L_0) \times \mathbb{R} \times \mathbb{R}^+$ is as follows:

$$\frac{dQ_0}{dy} = \lambda f(U_0) \tag{4.32}$$

$$\frac{d^2 U_0}{dy^2} + Q_0 \frac{dU_0}{dy} + f(U_0) = 0 \tag{4.33}$$

$$\text{and } W_0 = U_0, \tag{4.34}$$

subject to the initial conditions

$$Q(0) = \frac{dU}{dy}(0) = 0 \text{ and } U_0(0) = U_{b0} \tag{4.35}$$

and the matching conditions

$$U_0(L_0) = u_c \tag{4.36}$$

$$\text{and } \frac{dU_0}{dy}(L_0) + (u_c - u_a) Q_0(L_0) = 0. \tag{4.37}$$

The existence of a solution of the free boundary problem defined by (4.32–4.37) is a necessary condition for the existence of a branch of solutions bifurcating from infinity as $\mu \to 0$. The following theorem establishes this result, for the case $f(U) \propto U^2$ which occurs in practice.

THEOREM 4.8. *For $0 < \lambda < \lambda_c$ and $f(U) \propto U^2$ there exists an odd number, greater than or equal to one, of solutions of the free boundary problem defined by equations (4.32–4.37).*

PROOF: The proof employs a shooting technique. The details may be found in Stuart [8].

□

5. Existence Theory for (P2).

In this section we discuss the existence theory for the steady solutions of problem (P2). It may be shown that all steady solutions $V(y)$ of (3.5) and (3.6) are symmetric and so they satisfy

$$\frac{d^2V}{dy^2} + \lambda H(V-1)g(V) = 0 \tag{5.1}$$

$$\text{and } \frac{dV}{dy}(0) = V(1) = 0, \tag{5.2}$$

where $g(V)$ is a strictly positive $C^2(1,\infty)$. Note that we have redefined g by placing a parameter λ outside it. For the purposes of this section we assume that $g(1) \neq 0$. The trivial solution $V \equiv 0$ satisfies (5.1) and (5.2). However, by virtue of the Heaviside step function appearing in the forcing term, all other solutions satisfy $\|V\|_\infty > 1$. Thus, as for the steady travelling wave solutions of problem (P1), bifurcation from the trivial solution is precluded. Hence we develop an analogous approach to that in section 4 and apply it to the boundary value problem defined by equations (5.1) and (5.2).

If we seek solutions of (5.1) and (5.2) satisfying $\|V\|_\infty > 1$ and define $y = s$ to be the unique point $s \in (0,1)$ such that $V(s) = 1$ then such solutions may be shown to be in one-to-one correspondence with solutions of the following free boundary problem: find $(V(y), s) \in C^2(0,s) \times [0,1)$ with $V(y) \geq 1$ for $y \in (0,s)$ such that

$$\frac{d^2V}{dy^2} + \lambda g(V) = 0, \tag{5.3}$$

$$\text{and } \frac{dV}{dy}(0) = 0, \quad V(s) = 1 \text{ and } \frac{dV}{dy}(s) = -\frac{1}{1-s}. \tag{5.4}$$

Rescaling the independent variable y and the parameter λ by setting

$$z = y/s \text{ and } \bar{\lambda} = \lambda s \tag{5.5}$$

we obtain, from equations (5.3) and (5.4), the associated problem (FBP3), for $(V(z), s) \in C^2(0,1) \times \mathbb{R}$, namely

$$\frac{d^2V}{dz^2} + \bar{\lambda} s g(V) = 0 \tag{5.6}$$

$$\text{and } \frac{dV}{dz}(0) = 0, \quad V(1) = 1 \text{ and } \frac{dV}{dz}(1) = -\frac{s}{1-s}. \tag{5.7}$$

THEOREM 5.1. *Solutions $(V(z), s)$ of (FBP3) which satisfy $s \in (0,1)$ and $V(z) \geq 1$ for $z \in [0,1]$ are in a one-to-one correspondence with solutions $(V(y), s)$ of the original boundary value problem defined by (5.1) and (5.2).*

PROOF: Provided that $s \in (0,1)$ the mappings between the two problems are all bijections.

□

Thus we examine solutions of (FBP3). Clearly the trivial solution $V \equiv 1$ and $s = 0$ satisfies (FBP3). This solution exists for all values of the parameter $\bar{\lambda}$, but does not correspond to a solution of equations (5.1) and (5.2) since it does not satisfy $s > 0$. However, by applying local bifurcation theory to (FBP3) we may determine the possible location of bifurcation points for non-trivial branches of solutions which satisfy $s > 0$.

THEOREM 5.2. *There exists a branch of non-trivial solutions to (FBP3) bifurcating from the trivial solution $V \equiv 1$ and $s = 0$, at $\lambda = 1/g(1)$.*

PROOF: We define the function $W(z)$ by $W(z) = V(z) - 1$. We study solutions of (FBP3) which have the property that s is bounded away from unity by a finite amount. This will necessarily be the case for non-trivial solutions which bifurcate from the trivial solution $(W(z), s) = (0, 0)$. In this case (FBP3) may be written in the form

$$M(\lambda, x) = Bx - \bar{\lambda}Ax + N(\bar{\lambda}, x) = 0$$

where

$$B = \begin{pmatrix} \frac{d^2}{dz^2} & 0 \\ \frac{d}{dz}\Big|_{z=1} & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -g(1) \\ 0 & 0 \end{pmatrix}$$

$$\text{and } N = \begin{pmatrix} \bar{\lambda}s[g(W+1) - g(1)] \\ \frac{s^2}{1-s} \end{pmatrix}$$

for

$$x \in X = \{(W, s) : W \in C^2[0,1], \frac{dW}{dz}(0) = 0, W(1) = 0, \text{ and } s \in \mathbb{R}\}.$$

M defines a nonlinear mapping between the Banach spaces $\mathbb{R} \times X$ and Z, where $Z = \{(\tilde{W}, \tilde{s}) : \tilde{W} \in C[0,1] \text{ and } \tilde{s} \in \mathbb{R}\}$. We equip $\mathbb{R} \times X$ with the norm

$$\left\|(\lambda, W, s)\right\| = |\lambda| + \sum_{j=0}^{2} \sup_{z\in[0,1]} |W^j(z)| + |s|$$

(where ^{j} denotes the j-th derivative with respect to z) and Z with the norm

$$\left\|(\tilde{W}, \tilde{s})\right\| = \sup_{z\in[0,1]} |\tilde{W}(z)| + |\tilde{s}|.$$

In order that we may apply the standard theorems of local bifurcation theory [**1**] we require M to be $C^2(\mathbb{R} \times X, Z)$. However, in the neighborhood of $s = 1$ this is not the case since the nonlinear operator N includes the term $\frac{s^2}{1-s}$. Thus, in practice, we define a C^2 extension of the operator N by modifying N for $s \geq 1 - \delta$ where δ is a small strictly positive number. However, since the theorem concerns the local behavior of solutions in the neighborhood of the trivial solution $(W(z), s) = (0,0)$ we do not spell out the details of the extension. With the appropriate extension of N, M defines a mapping $\in C^2(\mathbb{R} \times X, Z)$.

Let $y = (0,0)$, the trivial solution. Then $N(\bar{\lambda}, y) = 0$. Also

$$D_x N(\bar{\lambda}, x) = \begin{pmatrix} \bar{\lambda} s \frac{dg}{dV}(W+1) & \bar{\lambda}[g(W+1) - g(1)] \\ 0 & \frac{2s-s^2}{(1-s)^2}. \end{pmatrix}$$

Thus $D_x N(\bar{\lambda}, y) = 0$. Hence, by Theorem 5.3 in Chapter 5 of [**1**], we deduce that if λ_0 is a simple eigenvalue of the pair (B, A) then (λ_0, y) is a bifurcation point of $M(\bar{\lambda}, x) = 0$. Thus we determine the location of simple eigenvalues of the pair (B, A); if we denote the corresponding eigenvectors by $x_0 = (\phi, \eta)$ then they satisfy

$$\frac{d^2\phi}{dz^2} + \eta\bar{\lambda}g(1) = 0, \tag{5.8}$$

$$\text{and } \frac{d\phi(0)}{dz} = 0, \quad \phi(1) = 0 \text{ and } \frac{d\phi}{dz} = -\eta. \tag{5.9}$$

Integration of the eigenvalue problem defined by (5.8–5.9) shows that $(B - \bar{\lambda}A)x_0 = 0$ has a nontrivial solution

$$x_0 = (\phi_0, \eta_0), \text{ where } \phi_0 = \frac{\eta_0}{2}(1 - z^2), \quad \eta_0 \in \mathbb{R}\backslash\{0\},$$

if and only if

$$\bar{\lambda} = \lambda_0 = 1/g(1).$$

A necessary and sufficient condition for the eigenvalue λ_0 to be simple is that the equation

$$(B - \lambda_0 A)x = Ax_0 \tag{5.10}$$

does not possess a solution. Defining $x = (\Phi, \gamma)$ equation (5.10) may be written as

$$\frac{d^2\Phi}{dz^2} + \gamma = -g(1)\eta \tag{5.11}$$

together with boundary conditions

$$\frac{d\Phi(0)}{dz} = 0, \quad \Phi(1) = 0 \text{ and } \frac{d\Phi(1)}{dz} = -\gamma. \tag{5.12}$$

Integration from $z = 0$ to $z = 1$ shows that equations (5.11–5.12) do not possess a solution unless $g(1) = 0$. However this case is excluded at the beginning of the section and thus we have proved that λ_0 is a simple eigenvalue of the pair (B, A). This completes the proof.

□

In the neighborhood of the bifurcation point we deduce from Theorem 5.3 in Chapter 5 of [**1**] that the solutions of (FBP3) satisfy

$$\bar{\lambda} = 1/g(1) - O(\varepsilon), \tag{5.13}$$

$$V(z) = 1 + \frac{\varepsilon}{2}(1 - z^2) + O(\varepsilon^2) \tag{5.14}$$

$$\text{and } s = \varepsilon + O(\varepsilon^2) \tag{5.15}$$

where $\varepsilon \ll 1$ is a measure of proximity to the bifurcation point. The value $\varepsilon = 0$ corresponds to the bifurcation point itself. Clearly for $0 < \varepsilon \ll 1$ we will have $0 < s \ll 1$ and hence, by Theorem 5.1, these solutions correspond to genuine solutions of the original boundary value problem defined by equations (5.1–5.2). By virtue of the transformation (5.5) we deduce from (5.13) and (5.15) that in the neighborhood of the bifurcation point in the original problem

$$\lambda = 1/\varepsilon g(1) + O(1)$$

and thus that since bifurcation corresponds to $\varepsilon \to 0$, it occurs from $\lambda = \infty$. Furthermore, in terms of the original independent variable y, the solution (5.14) corresponds to a solution

$$V(y) = \frac{1+y}{1-\varepsilon} + O(\varepsilon^2), \text{ for } -1 < y < -s$$
$$V(y) = 1 + \frac{\varepsilon}{2}(1 - y^2/\varepsilon^2) + O(\varepsilon^2), \text{ for } -s < y < s$$
$$\text{and } V(y) = \frac{1-y}{1+\varepsilon} + O(\varepsilon^2), \text{ for } s < y < 1.$$

Taking the limit $\varepsilon \to 0$, which corresponds to approaching the bifurcation point at $\lambda = \infty$, we obtain

$$\lim_{\varepsilon \to 0} V(y) = 2g(y;0)$$

where $g(y;z)$ is the Greens function for the problem, satisfying

$$\frac{d^2g}{dy^2} + \delta(y-z) = 0$$

and

$$g(\pm 1) = 0.$$

The coefficient 2 appears so that the solution satisfies the condition that

$$\lim_{\lambda \to \infty} V(0) = 1.$$

Similar results on the approach of solutions to Greens functions in certain parameter limits have been found by Keady and Norbury [**4**].

6. Summary.

In this paper we have presented a constructive approach to the existence theory for two steady state problems (P1) and (P2) to which local bifurcation theory is not directly applicable. Although both problems possess a trivial solution the appearance of a Heaviside step function in the reaction terms (which consequently have a switch-like behavior) prevents bifurcation from the trivial solutions.

However, for both (P1) and (P2), the equations are linear outside a finite interval of unknown length, in which the reaction terms are non-zero. Thus, by solving the equations explicitly outside this region the problems may both be reformulated as two-point free boundary problems. A rescaling, which is problem dependent, renders both problems in a form to which local bifurcation theory is directly applicable. In particular we have proved that for problem (P2) bifurcation occurs from a parameter value at infinity and that the solution approaches a multiple of the Greens function for the domain as that parameter approaches infinity.

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